Topic: Impossibility of Integer solutions for simultaneous equations given below

$$(p+q) = (r+s) \quad and$$

$$p^k + q^k = r^k + s^k$$

$$For\ k = 2,3,4,5$$

$$But\ for\ \ k = 6\ we\ will\ consider\ simultaneous\ equations$$

$$(p^2+q^2) = (r^2+s^2) \quad and$$

$$p^6 + q^6 = r^6 + s^6$$

Eventhough due to "Extended Euler's Conjecture states that

$(m+n-k) > 0$ *or* $= 0$, *we have attempted to demonstrate solvability for* $k = 5\ \&\ 6$ *because there could be exceptions to the conjecture.*

In this context (m, n) *are number of terms in the equation* & (k) *is the degree.*

In our case $m = n = 2$

Consider the well known historical parametric solutions given below:

Equation $p^2 + q^2 = r^2 + s^2$ ------- (I)

Solution is:

$$p = ad - bc, \qquad q = ab + cd, \qquad r = ad + bc, \qquad s = ab - cd$$

Equation:

$$(p^2 - pq + q^2) = (r^2 - rs + s^2) \ -----------------(II)$$

Solution is

$$p = b + c + d, \qquad q = d - a, \qquad r = a + c + d, \qquad s = c - b$$

Where ad=bc

For k=2 , simultaneous equations

$$(p+q) = (r+s) \ --(A)$$

$$p^2+q^2 = r^2+s^2 \quad ---(B)$$

$$Using\ \ parametric\ Solution\ from\ equation\ (I)\ at\ the\ top\ we\ see\ that$$

$$The\ 2nd\ Part\ (B) is\ already\ satisfied.\ We\ substitute\ value\ of$$

$$(p,q,r,s) in\ the\ Ist\ part\ (p=q)=(r+s)\ and\ we\ get,$$

$$(ad-bc+ab+cd)=(ad+bc+ab-cd)$$

$$Simplyfing\ we\ get \quad 2bc=2cd$$

$$Which\ means\ b=d$$

$$Substituting\ b=d\ in\ the\ solution\ for\ (p,q,r,s) we\ get$$

$$p=b(a-c), \qquad q=b(a+c), \qquad r=b(a+c), \qquad s=b(a-c)$$

$$This\ means\ we\ get\ a\ trivial\ solution.$$

$$Hence\ for\ k=2, \qquad Integer\ solutions\ for\ simultaneous\ equations$$

$$(p+q) = (r+s) \quad --(A) and$$

$$p^2+q^2 = r^2+s^2 \quad ---(B)\ is\ not\ possible.$$

For K=3, simultaneous equations

$$(p+q) = (r+s) \quad --(A)$$

$$p^3+q^3 = r^3+s^3 \quad ---(B)$$

$$Part\ (B)\ \ becomes\ (p+q)(p^2-pq+q^2)=(r+s)(r^2-rs+s^2)$$

$$Using\ parametric\ Solution\ from\ equation\ (II)\ at\ the\ top\ we\ see\ that$$

$$(p^2-pq+q^2)=(r^2-rs+s^2)$$

Hence for part (B) to be true than $(p+q) = (r+s)$

So substituting values of (p,q,r,s) we get

$$b+c+d+d-a=a+c+d+c-b$$

$$Simplyfing\ we\ get\ 2(a-b)=(d-c).\ Using\ the\ condition\ ad=bc\ we\ get$$

$$d=-(2b) and\ c=-(2a).$$

$$Substituting\ values\ of\ (c,d) from\ above\ we\ get$$

$$p=-2a-b, q=-a-2b, \qquad r=-a-2b, s=-2a-b$$

$$This\ means\ we\ get\ a\ trivial\ solution.$$

$$Hence\ for\ k=3, \qquad Interger\ solutions\ for\ simultaneous\ equations$$

$$(p+q)=\ (r+s)\ \ --(A) and$$

$$p^3+q^3=\ r^3+s^3\ \ ---(B)\ is\ not\ possible.$$

For K=4, simultaneous equations

$$(p+q)=\ (r+s)\ \ --(A)$$

$$p^4+q^4=\ r^4+s^4\ \ ---(B)$$

We have parametric solution For (b) as:

$$p=(a+3a^2-2a^3+a^5+a^7)$$

$$q=(1+a^2-2a^4-3a^5+a^6)$$

$$r=(a-3a^2-2a^3+a^5+a^7)$$

$$s=(1+a^2-2a^4+3a^5+a^6)$$

Second part (B) is already satisfied. Hence in part (A) we substitute the values of (p,q,r,s) we get

$$(a+3a^2-2a^3+a^5+a^7)+(1+a^2-2a^4-3a^5+a^6) = (\ a-3a^2-2a^3+a^5+a^7)+(1+a^2-2a^4+3a^5+a^6)$$

We get after simplyfication 6a^2*(a^3-1) = 0 & we get (a=1)

But for a=1, Equation (A) is trivial.

$$Hence\ for\ k=4, \qquad Integer\ solutions\ for\ simultaneous\ equations$$

$$(p+q)=\ (r+s)\ \ --(A)\ and$$

$$p^4 + q^4 = r^4 + s^4 \quad --- (B) \; is \; not \; possible.$$

For k=5, simultaneous equations:

$$(p+q) = (r+s) \quad --(A)$$

$$p^5 + q^5 = r^5 + s^5 \quad --- (B)$$

We have identities given below:

$(p+q)^5 = p^5 + q^5 + 5pq(p+q)(p^2 + pq + q^2)$ -----(I)

$(r+s)^5 = r^5 + s^5 + 5rs(r+s)(r^2 + rs + s^2)$ ---------(II

We have parametric solution (arrived at by Ajai Choudhry & Wroblewski)
For the equation, [$q(p+q)(p^2 + pq + q^2)$ = $rs(r+s)(r^2 + rs + s^2)$]

Where;

$$p = m(m+n)BD$$
$$q = 2mn(3m^{10} + n^{10})$$
$$r = n(m+n)AD$$
$$s = n(m-n)BC$$

Where

$$A = -n^{10} + 4n^5m^5 + m^{10}$$
$$B = -n^{10} - 4n^5m^5 + m^{10}$$
$$C = n^4 + n^3m + n^2m^2 + nm^3 + m^4$$
$$D = n^4 - n^3m + n^2m^2 - nm^3 + m^4$$

Since $pq(p+q)(p^2 + pq + q^2)$ =
$rs(r+s)(r^2 + rs + s^2) \; is \; satisfied \; due \; to \; solution \; given \; above$
Subtracting Equations (I) & (II) we get

$(p+q)^5 - p^5 - q^5 = (r+s)^5 - r^5 - s^5$

$$Hence \;\; p^5 + q^5 = r^5 + s^5$$
$$when \; (p+q) = (r+s) \; ------(III)$$

$$Substituting \; value \; of \; (p,q,r,s) \; in \; equation \; (III) \; and \; simplifying \; we \; get$$

$$(n^4 - mn^3 - m^2n^2 - m^3n + m^4) = 0 \quad ---------------(IV)$$

$$(n^{10} + 2n^8m^2 + 2n^7m^3 - 2n^6m^4 + 2m^5n^5 - 2n^3m^7 + 2n^2m^8 - 2nm^9 + m^{10}) = 0 \;----------\; (V)$$

$$We\ ignore\ the\ (m-n)\ factor\ since\ m = n\ gives\ a\ trivial\ solution$$

$$Equation\ (IV)\ is\ equivalent\ to\ (m^2 + n^2)^2 = mn(m^2 + 3mn + n^2)$$
$$Letting\ m = u\^{}2\ \&\ n = v\^{}2\ we\ get\ (u\^{}4 + v\^{}4) = uv(u^4 + 3u^2v^2 + v^4)\^{}(1/2)$$

And since $(u^4 + 3u^2v^2 + v^4)$ *is not a square for integers* (u, v) *hence equation* (IV) *does not have integral solution*

Similarly Equation (V) *does not have Integral solution.*

hence equation (III) *when* $(p+q) = (r+s)$ *cannot be satisfied.*

Hence $(p^5 + q^5 = r^5 + s^5)$ does not have integer solutions

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For k=6, simultaneous equations:

$$(p^2 + q^2) = (r^2 + s^2) \;--(A)$$

$$p^6 + q^6 = r^6 + s^6 \;---\; (B)$$

$$Where, \quad p = ad - bc, \quad q = ab + cd, \quad r = ad + bc, \quad s = ab - cd$$

Equation (B) *is equivalent to*

$$(p^2 + q^2)(p^4 - p^2q^2 + q^4) = (r^2 + s^2)(r^4 - r^2s^2 + s^4)$$

Since $(p^2 + q^2) = (r^2 + s^2)$ *is equal due to substitution given above*

Hence $(p^4 - p^2q^2 + q^4) = (r^4 - r^2s^2 + s^4)$

But the later equation is equivalent to $(p^2 + q^2)^2 - 3p^2q^2 = (r^2 + s^2)^2 - 3r^2s^2$

Which means due to equation (A) $3p^2q^2 = 3r^2s^2$ *or* $pq = rs$

Substituting values of (p, q, r, s) *we get*

$$(ad - bc)(ab + cd) = (ad + bc)(ab - cd)$$

Solving we get (b=d)

Substituting (b=d) in the values of (p,q,r,s) we get

$$p = b(a - c), \qquad q = b(a + c), \qquad r = b(a + c), \qquad s = b(a - c)$$

Which is a trivial solution.

$$Subsequently\ \ equation\ \ (p^2 + q^2) = (r^2 + s^2)\ cannot\ be\ satisfied.$$

$Hence\ \ (p^6 + q^6 = r^6 + s^6)$ does not have integer solutions.

$$Hence\ for\ k = 6, \qquad Integer\ solutions\ for\ simultaneous\ equations$$

$$(p^2 + q^2) = \ (r^2 + s^2)\ \ --(A)\ and$$

$$p^6 + q^6 = \ r^6 + s^6\ \ ---(B)\ is\ not\ possible.$$
