

How did Euler solve the equation $xyz(x + y + z) = a$?

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Abstract

In this paper, we derived the parametric solution of Euler and Elkies in an elementary manner. In addition we proved there are infinitely many parametric solutions of Euler's and Elkies's family of solutions.

1. Introduction

According to Elkies[1], in 1749 Euler takes a look at $xyz(x + y + z) = a$ and says that he has found, with quite some effort. His parametric solution is as follows.

$$\begin{aligned}x &= \frac{6ast^3(at^4 - 2s^4)^2}{(4at^4 + s^4)(2a^2t^8 + 10as^4t^4 - s^8)} \\y &= \frac{3}{2} \frac{s^5(4at^4 + s^4)^2}{t(at^4 - 2s^4)(2a^2t^8 + 10as^4t^4 - s^8)} \\z &= \frac{2}{3} \frac{(2a^2t^8 + 10as^4t^4 - s^8)}{s^3t(4at^4 + s^4)}\end{aligned}$$

We don't know how he got his solution.

In 2014, Elkies looked for a solution, and he got some solutions using algebraic geometry. Simpler solution is as follows.

$$\begin{aligned}x &= \frac{(s^4 - 4a)^2}{2s^3(s^4 + 12a)} \\y &= \frac{2a(3s^4 + 4a)^2}{s^3(s^4 - 4a)(s^4 + 12a)} \\z &= \frac{s(s^4 + 12a)}{2(3s^4 + 4a)}\end{aligned}$$

In 2022, inspired by the problem posted on MathStackExchange[2], it is related to our problem, so we decided to work on elementary derivation of Euler's solution.

We proved that there are infinitely many parametric solutions of Euler's family of solutions.

In addition we derived Elkies's solution in an elementary manner and proved that there are infinitely many parametric solutions of Elkies's family of solutions.

Moreover we showed the small positive solutions table for $a < 100$.

Furthermore, parametric solution of $wxyz(w + x + y + z) = a$ was shown in Appendix.

An infinitely many solutions are generated from infinite order point of elliptic curve using group law. In this context, we call generated solutions are family of solutions of infinite order point.

2. Derive Euler's solution

According to Elkies, Euler found a parametric solution of $xyz(x + y + z) = a$ below.

$$\begin{aligned} x &= \frac{6ast^3(at^4 - 2s^4)^2}{(4at^4 + s^4)(2a^2t^8 + 10as^4t^4 - s^8)} \\ y &= \frac{3}{2} \frac{s^5(4at^4 + s^4)^2}{t(at^4 - 2s^4)(2a^2t^8 + 10as^4t^4 - s^8)} \\ z &= \frac{2}{3} \frac{(2a^2t^8 + 10as^4t^4 - s^8)}{s^3t(4at^4 + s^4)} \end{aligned}$$

We prove that Euler's family of solutions has an infinitely many parametric solutions where a is arbitrary.

Proof.

$$xyz(x + y + z) = a$$

We define Euler's solution form as follows.

$$\begin{aligned} x &= \frac{c_1 a s_1^t t_1^t A^2}{BC} \\ y &= \frac{c_2 s_2^s t_2^t B^2}{AC} \\ z &= \frac{c_3 s_3^s t_3^t C}{B} \end{aligned}$$

Taking $(s_1, s_2, s_3) = (1, 5, -3)$, $(t_1, t_2, t_3) = (3, -1, -1)$, $(c_1, c_2, c_3) = (6, \frac{3}{2}, \frac{2}{3})$, then

$$\begin{aligned} x &= \frac{6Ast^3A^2}{BC} \\ y &= \frac{3}{2} \frac{s^5B^2}{tAC} \\ z &= \frac{2}{3} \frac{C}{s^3tB} \end{aligned}$$

Hence

$$xyz(x + y + z) = \frac{a(36As^4t^4A^3 + 9s^8B^3 + 4C^2A)}{C^2B}$$

RHS of above equation must be a, then

$$36As^4t^4A^3 + 9s^8B^3 + 4C^2A = C^2B$$

Since C must be rational number then discriminant must be square number.

Let $U = \frac{A}{B}$ we get quartic curve

$$V^2 = -16at^4U^4 + 4at^4U^3 - 4s^4U + s^4$$

Quartic is birationally equivalent to an elliptic curve.

$$E : Y^2 - 4s^2YX + 8s^2at^4Y = X^3 - 4s^4X^2 + 64at^4s^4X - 256at^4s^8$$

E has a point $P(X, Y) = (4s^4, 16s^6 - 8s^2at^4)$.

Hence we can obtain a point $2Q(U) = \frac{at^4 - 2s^4}{4at^4 + s^4}$ from $2P(X, Y)$ using group law.

Then we obtain

$$(A, B, C) = (at^4 - 2s^4, 4at^4 + s^4, -s^8 + 10as^4t^4 + 2a^2t^8)$$

Finally, we obtain an Euler's solution.

$$x = \frac{6ast^3(at^4 - 2s^4)^2}{(4at^4 + s^4)(2a^2t^8 + 10as^4t^4 - s^8)}$$

$$y = \frac{3}{2} \frac{s^5(4at^4 + s^4)^2}{t(at^4 - 2s^4)(2a^2t^8 + 10as^4t^4 - s^8)}$$

$$z = \frac{2}{3} \frac{(2a^2t^8 + 10as^4t^4 - s^8)}{s^3t(4at^4 + s^4)}$$

Similarly, we can obtain other new solutions using group law with $P(X, Y) = (4s^4, 16s^6 - 8s^2at^4)$.

Hence Euler's family of solutions has an infinitely many parametric solutions.

For instance, we can obtain a new solution using $3P(X, Y)$ as follows.

$$x = \frac{18as^5t^3(-2s^4 + at^4)^2(-s^8 + 10at^4s^4 + 2a^2t^8)^2(a^2t^8 - 5s^8 + 14at^4s^4)}{(at^4 + s^4)(a^3t^{12} + 3s^4a^2t^8 + 111at^4s^8 + s^{12})(a^6t^{24} + 6s^4a^5t^{20} - 255s^8a^4t^{16} - 790a^3t^{12}s^{12} - 2253a^2t^8s^{16} - 264s^{20}at^4 + s^{24})}$$

$$y = \frac{1}{6} \frac{-(at^4 + s^4)^2(a^3t^{12} + 3s^4a^2t^8 + 111at^4s^8 + s^{12})^2(a^2t^8 - 5s^8 + 14at^4s^4)}{s^3t(-2s^4 + at^4)(-s^8 + 10at^4s^4 + 2a^2t^8)(a^6t^{24} + 6s^4a^5t^{20} - 255s^8a^4t^{16} - 790a^3t^{12}s^{12} - 2253a^2t^8s^{16} - 264s^{20}at^4 + s^{24})}$$

$$z = \frac{2s(a^6t^{24} + 6s^4a^5t^{20} - 255s^8a^4t^{16} - 790a^3t^{12}s^{12} - 2253a^2t^8s^{16} - 264s^{20}at^4 + s^{24})}{(a^2t^8 - 5s^8 + 14at^4s^4)t(at^4 + s^4)(a^3t^{12} + 3s^4a^2t^8 + 111at^4s^8 + s^{12})}$$

The proof is completed.

3. Derive Elkies's solution

According to Elkies, Elkies found a parametric solution of $xyz(x + y + z) = a$ below.

$$x = \frac{1}{2} \frac{(s^4 - 4a)^2}{s^3(s^4 + 12a)}$$

$$y = \frac{2a(3s^4 + 4a)^2}{s^3(s^4 - 4a)(s^4 + 12a)}$$

$$z = \frac{1}{2} \frac{s(s^4 + 12a)}{3s^4 + 4a}$$

We show Elkies's family of solutions has an infinitely many parametric solutions where a is arbitrary.

Proof.

$$xyz(x + y + z) = a$$

Taking

$$x = \frac{1}{2} \frac{A^2}{s^3 C}$$

$$y = \frac{2aB^2}{s^3 AC}$$

$$z = \frac{1}{2} \frac{sC}{B}$$

Hence

$$xyz(x + y + z) = \frac{1}{4} \frac{a(A^3 B + 4aB^3 + s^4 C^2 A)}{s^8 C^2}$$

RHS of above equation must be a , then

$$A^3 B + 4aB^3 + s^4 C^2 A - 4s^8 C^2 = 0 \tag{1}$$

Since C must be rational number then discriminant must be square number.

$$v^2 = -4(A - 4s^4)s^4 B(A^3 + 4aB^2)$$

Let $B = -A + 4s^4$ then

$$V^2 = A^3 + 4aB^2$$

$$= A^3 + 4aA^2 - 32s^4 aA + 64s^8 a \tag{2}$$

In order to find a parametrization for A, V , substitute $(A, V) = (s^4 + p, s^6 + qs^4 + rs^2)$ to (2). We obtain $(p, q, r) = (-4a, 0, 12a)$ then $(A, B, C) = (s^4 - 4a, 3s^4 + 4a, s^4 + 12a)$. Finally, we obtain Elkies's solution.

$$x = \frac{1}{2} \frac{(s^4 - 4a)^2}{s^3(s^4 + 12a)}$$

$$y = \frac{2a(3s^4 + 4a)^2}{s^3(s^4 - 4a)(s^4 + 12a)}$$

$$z = \frac{1}{2} \frac{s(s^4 + 12a)}{3s^4 + 4a}$$

From (2), we define elliptic curve E

$$E : V^2 = A^3 + 4aA^2 - 32s^4 aA + 64as^8$$

Since E has a point $P(A, V) = (s^4 - 4a, s^6 + 12as^2)$,

we can obtain a point $2P(A) = \frac{1s^{16} - 464s^{12}a + 1632s^8a^2 + 768a^3s^4 + 256a^4}{4s^4(s^4 + 12a)^2}$.

According to Nagell-Lutz theorem, the point $P(A, V)$ is not a point of finite order. Hence Elkies's family of solutions has an infinitely many parametric solutions.

We can obtain new solutions using group law with $P(A, V) = (s^4 - 4a, s^6 + 12as^2)$. For instance, we can obtain a new solution using $2P(A, V)$ as follows.

We obtain

$$A = \frac{1}{4} \frac{(-4a + s^4)(s^{12} - 460s^8a - 208a^2s^4 - 64a^3)}{s^4(s^4 + 12a)^2}$$

$$B = \frac{1}{4} \frac{(-4a + 5s^4)(4a + 3s^4)(s^8 + 56s^4a + 16a^2)}{s^4(s^4 + 12a)^2}$$

$$C = \frac{-1}{8} \frac{(-16a^2 - 32s^4a + s^8)(s^{16} + 1136s^{12}a - 928s^8a^2 + 1792a^3s^4 + 256a^4)}{s^8(s^4 + 12a)^3}$$

$$x = \frac{-1}{4} \frac{(-4a + s^4)^2(s^{12} - 460s^8a - 208a^2s^4 - 64a^3)^2}{s^3(s^4 + 12a)(-16a^2 - 32s^4a + s^8)(s^{16} + 1136s^{12}a - 928s^8a^2 + 1792a^3s^4 + 256a^4)}$$

$$y = \frac{-4(s^4 + 12a)s(-4a + 5s^4)^2(4a + 3s^4)^2(s^8 + 56s^4a + 16a^2)^2a}{(s^{16} + 1136s^{12}a - 928s^8a^2 + 1792a^3s^4 + 256a^4)(-16a^2 - 32s^4a + s^8)(-4a + s^4)(s^{12} - 460s^8a - 208a^2s^4 - 64a^3)}$$

$$z = \frac{-1}{4} \frac{(-16a^2 - 32s^4a + s^8)(s^{16} + 1136s^{12}a - 928s^8a^2 + 1792a^3s^4 + 256a^4)}{s^3(-4a + 5s^4)(4a + 3s^4)(s^8 + 56s^4a + 16a^2)(s^4 + 12a)}$$

The proof is completed.

4. Solution table

Small positive solutions by brute force search with $a < 100$.

Table 1: Solutions of $xyz(x + y + z) = a$

a	x	y	z
1	3/2	4/3	1/6
2	5/2	5/6	4/15
3	1	1	1
4	7/2	36/35	7/30
5	4	1/2	1/2
6	2	3/2	1/2
7	10/3	21/20	5/12
8	2	1	1
9	2	2	1/2
10	5/2	4/3	2/3
11	2	11/6	2/3
12	5/3	27/20	5/4
13	3/2	3/2	4/3
14	2	4/3	7/6
15	3	1	1
16	3	8/3	1/3
17	5/2	10/7	34/35
18	4	3/2	1/2
19	3	3	1/3
20	2	2	1
21	5/2	5/2	3/5
22	9/2	25/6	2/15
23	5	9/10	23/30
24	4	1	1
25	4	9/4	5/12
26	4	2	1/2
27	9/4	25/12	16/15
28	3	7/3	2/3
29	4	29/20	4/5
30	4	15/4	1/4
31	8/3	31/12	3/4
32	5	5/3	8/15
33	2	2	3/2
34	7/2	7/3	9/14
35	3	3/2	4/3
36	3	2	1
37	20	37/15	1/30
38	6	12/7	19/42
39	4	3/2	1
40	3	3	2/3
41	3	25/12	16/15
42	15/4	7/5	5/4
43	9/2	2	2/3
44	7	7/4	11/28
45	4	3	1/2
46	8	25/12	4/15
47	25/6	32/15	3/4
48	2	2	2
49	6	3/2	2/3
50	5	5/2	1/2
51	4	17/12	4/3
52	6	26/15	3/5
53	7/2	7/2	4/7
54	5	8/5	9/10

Table 1: Solutions of $xyz(x + y + z) = a$

a	x	y	z
55	5	11/3	1/3
56	4	2	1
57	16/5	25/8	4/5
58	20/3	6/5	5/6
59	5	49/30	20/21
60	8	3/2	1/2
61	32/3	9/2	1/12
62	9/4	25/12	31/15
63	3	3	1
64	9	1	2/3
65	3	13/6	3/2
66	5/2	5/2	8/5
67	15/2	32/15	5/12
68	4	4	1/2
69	13/4	13/4	23/26
70	6	35/6	1/6
71	10	49/20	8/35
72	16	25/8	3/40
73	6	6	1/6
74	15/2	10/3	4/15
75	4	5/2	1
76	7/2	18/7	7/6
77	5	16/5	11/20
78	4	13/4	3/4
79	8	25/12	9/20
80	5	2	1
81	9/2	4	1/2
82	10/3	5/2	41/30
83	5	5/2	4/5
84	3	2	2
85	9	16/9	17/36
86	3	8/3	3/2
87	9	25/3	1/15
88	3	3	4/3
89	27/2	2/3	2/3
90	4	2	3/2
91	4	9/4	4/3
92	5	23/5	2/5
93	11/2	11/2	3/11
94	9/2	2	4/3
95	5	4	1/2
96	4	3	1
97	10/3	49/20	45/28
98	7/2	7/2	1
99	5/2	5/2	11/5

5. Final remarks

We used the Euler's solution form, combination of (A, B, C) to derive Euler's parametric solution. So, we think there must be parametric solutions other than Euler's form. The same could be said of Elkies case in a similar way. Furthermore, it might be interesting to construct the parametric solutions using (A, B, C, D) .

Appendix

We consider the extension of $xyz(x+y+z) = a$ and show $wxyz(w+x+y+z) = a$ has infinitely many parametric solutions where a is arbitrary. The problem $wxyz(w+x+y+z) = 1$ was posted on mathoverflow.net[3].

An equation $wxyz(w+x+y+z) = a$ has an infinitely many parametric solutions where a is arbitrary.

Proof.

$$wxyz(w+x+y+z) = a$$

$$\begin{aligned} w &= \frac{ac_1t^{t_1}}{ABC} \\ x &= \frac{c_2Bt^{t_2}}{A} \\ y &= \frac{c_3At^{t_3}}{C} \\ z &= \frac{c_4Ct^{t_4}}{B} \end{aligned}$$

Hence

$$wxyz(w+x+y+z) = \frac{c_1t^{t_1}c_2t^{t_2}c_3t^{t_3}c_4t^{t_4}(ac_1t^{t_1} + c_2B^2t^{t_2}C + c_3A^2t^{t_3}B + c_4C^2t^{t_4}A)}{A^2B^2C^2}$$

RHS of above equation must be a, then

$$\begin{aligned} &(c_1c_2c_3c_4^2t^{t_1+t_2+t_3+2t_4}A - A^2B^2)C^2 \\ &+ c_1c_2^2c_3c_4t^{t_4+t_3+t_1+2t_2}B^2C \\ &+ ac_1^2c_2c_3c_4t^{t_4+t_3+t_2+2t_1} + c_1c_2c_3^2c_4t^{t_1+t_2+t_4+2t_3}A^2B = 0 \end{aligned}$$

Since C must be rational number then discriminant must be square number.

$$\begin{aligned} v^2 &= 4c_1c_2c_3^2c_4t^{t_4+t_2+t_1+2t_3}B^3A^4 \\ &- 4c_1^2c_2^2c_3^3c_4^3Bt^{3t_4+3t_3+2t_2+2t_1}A^3 \\ &+ a4c_1^2c_2c_3c_4t^{t_4+t_3+t_2+2t_1}B^2A^2 \\ &- a4c_1^3c_2^2c_3^3c_4^3t^{3t_4+2t_3+2t_2+3t_1}A \\ &+ c_1^2c_2^4c_3^2c_4^2B^4t^{2t_4+2t_3+4t_2+2t_1} \end{aligned}$$

Obviously, quartic is birationally equivalent to an elliptic curve.

For instance, take $(c_1, c_2, c_3) = (1, 1, 1)$, $(t_1, t_2, t_3) = (1, 1, 1)$, $B = 1$, then

$$v^2 = 4t^5A^4 - 4t^{10}A^3 + 4t^5aA^2 - 4t^{10}aA + t^{10}$$

Quartic is transformed to an elliptic curve below.

$$E : Y^2 - 4t^5aYX - 8t^{15}Y = X^3 + (4t^5a - 4t^{10}a^2)X^2 - 16t^{15}X - 64t^{20}a + 64t^{25}a^2$$

E has a point $P(X, Y) = (-4t^5a + 4t^{10}a^2, -16t^{10}a^2 + 16t^{15}a^3 + 8t^{15})$.

We obtain

$$2P(X) = \frac{4((-a^2 - 2a^5 + a^8)t^{20} + (-2a^4 - a - 4a^7)t^{15} + (1 + 6a^6 + 6a^3)t^{10} + (-2a^2 - 4a^5)t^5 + a^4)}{(2a^3 + 1)t^5 - 2a^2}$$

According to Nagell-Lutz theorem, the point $P(X, Y)$ is not a point of finite order, hence we can obtain infinitely many parametric solutions.

Thus we can obtain a quartic point

$$2Q(A) = \frac{-t^5(-2a^2 + 2t^5a^3 + t^5)}{t^{10}a^4 - 2t^5a^3 - t^5 + a^2}$$

Then we obtain (A, C)

$$A = \frac{-t^5(-2a^2 + 2t^5a^3 + t^5)}{t^{10}a^4 - 2t^5a^3 - t^5 + a^2}$$

$$C = \frac{t^{10} + a - 2t^5a^2 + t^{10}a^3}{t^{10}a^2 - 1}$$

Finally we obtain (w, x, y, z)

$$w = \frac{-a(t^{10}a^4 - 2t^5a^3 - t^5 + a^2)(t^5a + 1)(t^5a - 1)}{t^4(-2a^2 + 2t^5a^3 + t^5)(t^{10} + a - 2t^5a^2 + t^{10}a^3)}$$

$$x = \frac{-(t^{10}a^4 - 2t^5a^3 - t^5 + a^2)}{t^4(-2a^2 + 2t^5a^3 + t^5)}$$

$$y = \frac{-t^6(-2a^2 + 2t^5a^3 + t^5)(t^5a + 1)(t^5a - 1)}{(t^{10}a^4 - 2t^5a^3 - t^5 + a^2)(t^{10} + a - 2t^5a^2 + t^{10}a^3)}$$

$$z = \frac{(t^{10} + a - 2t^5a^2 + t^{10}a^3)t}{(t^5a + 1)(t^5a - 1)}$$

In this way we can obtain infinitely many parametric solutions.
The proof is completed.

References

- [1] Noam Elkies, https://abel.math.harvard.edu/%7Eelkies/euler_14t.pdf
- [2] MathStackExchange, <https://math.stackexchange.com/questions/4496520/solving-the-diophantine-system-pqr-a4-pq>
- [3] Mathoverflow.net, <https://mathoverflow.net/questions/428053/equation-wxyzwxyz-1-in-mathbbq-4>