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Parametric solutions to equation $(pa^n + qb^n = pc^n + qd^n)$ where 'n' stands for degree 2,3,4,5,6,7,8 & 9

Authors: *Seiji Tomita*[1] *and Oliver Couto*[2]

1 Number theory theorist, Tokyo, Japan
Email: fermat@m15.alpha-net.ne.jp

2 University of Waterloo, Ontario, Canada
Email: samson@celebrating-mathematics.com

ABSTRACT

Historically equation $(pa^n + qb^n + rc^n = pu^n + qv^n + rw^n)$ has been studied for degree 2,3,4 etc., and equation $(pa^n + qb^n = pc^n + qd^n)$ herein called equation (1) has been published for n=4 ,p=1,q=4 (Ref.no. 1) by Ajai Choudhry. Also Tito Piezas & others has discussed about equation (1) (Ref. no. 3 & 2) . While Ref. no. (1, 2 & 3) deals with equation no. (1) for degree n=4 this paper has provided parametric solutions for degree n=2,3,4,5,6,7,8 & 9. Also there are instances in this paper where parametric solutions have been arrived at using different methods.

Keywords: Diophantine equations, Equal sums of powers, pure mathematics.

Parametric solutions to equation :

$$(pa^n + qb^n) = (pc^n + qd^n) \;----(1)$$

First we consider:

Degree two

For n=2 In equation (1) above;

We get $(pa^2 + qb^2 = pc^2 + qd^2)$

Hence we have after simplification:

$$\frac{p}{q} = \frac{d^2 - b^2}{a^2 - c^2}$$

We take p = $d^2 - b^2$ & q = $a^2 - c^2$

Let a=u+v, b=v-w, c=u-v, d=v+w
By resolving above polynomial into factors, we get {p,q}.
(p,q)=(w,u)

Hence we get the Identity given below:

$$w(u+v)^2 + u(v-w)^2 = w(u-v)^2 + u(v+w)^2$$

Let: (u,v,w)=(5,3,2)

(p,q,a,b,,c,d)=(2,5,8,1,2,5)

We get:

$$2(8)^2 + 5(1)^2 = 2(2)^2 + 5(5)^2$$

Solution to: $pa^2 + qb^2 = pc^2 + qd^2$ where (p,q)=(1,3),

We have:

$$(1)a^2 + (3)b^2 = (1)(c)^2 + (3)d^2$$

The above equation has numerical solution (a,b,c,d)=(1,5,7,3)

Using the substitution: (a,b,c,d)=(1+t,5+t,7+kt,3+t) we get after simplifying

$$t = (-\frac{14}{k+1})$$

Substituting value of 't' in (a,b,c,d) we get the below mentioned parametric equation:

$$(1)(k-13)^2 + 3(5k-9)^2 = (1)(7k-7)^2 + (3)(3k-11)^2$$

For k=3 we have numerical solution as:

$$(1)5^2 + 3(3)^2 = (1)(7)^2 + (3)(1)^2$$

(Next): Degree (n=3)

There is a parametric solution for:

$$pa^3 + qb^3 = pc^3 + qd^3$$

$$a = u^2(qv^6 + pu^6)$$

$$b = (-2pu^6 + qv^6)$$

$$c = v^2(qv^6 + pu^6)$$

$$d = -(2qv^6 - pu^6)$$

a, b, p, q: arbitrary

$$pa^3 + qb^3 = pc^3 + qd^3 \dots\dots\dots\dots..(1)$$

$$let\ a = u^2, \quad b = t + v^2, \ c = mt + u^2, \ d = v^2 ----(2)$$

$$(q - pm^3)t^3 + (-3pu^2m^2 + 3qv^2)t^2 + (-3pu^4m + 3qv^4)t = 0$$

$$Then\ we\ obtain, \quad m = \frac{qv^4}{pu^4} \quad \&$$

$$t = \frac{-3v^2pu^6}{qv^6 + pu^6}.$$

Substitute m and t to (2), and we obtain a solution.

Case. (p,q,v)=(2,3,1) (After removing common factors) we get parametric form:

$$a = u^2(3 + 2u^6)$$
$$b = (-4u^6 + 3)$$
$$c = 2u^2(u^6 - 3)$$
$$d = (3 + 2u^6)$$

$$2(u^2(3 + 2u^6))^3 + 3(-4u^6 + 3)^3 = 2(2u^2(u^6 - 3))^3 + 3(3 + 2u^6)^3$$

For u=2, p=2, q=3 we get numerical solution:

$$(2)(260)^3 + (3)(-253)^3 = (2)(488)^3 + (3)(131)^3$$

Alternate solution:

We have $(pa^3 + qb^3 = pc^3 + qd^3)$

We get:

$$\frac{p}{q} = \frac{d^3 - b^3}{a^3 - c^3}$$

Let a=u+v, b=v-w, c=u-v, d=v+w
By resolving above polynomial into factors, we get {p,q}.

$$p_3(u+v)^3 + q_3(v-w)^3 = p_3\ (u-v)^3 + q_3(v+w)^3$$

Where:

$$p_3 = w(3v^2 + w^2)$$
$$q_3 = v(3u^2 + v^2)$$

Let: (u,v,w)=(3,2,1)

(p,q,a,b,,c,d)=(31,126,5,1,1,3)

$$13(5)^3 + 62(1)^3 = 13(1)^3 + 62(3)^3$$

Adittional method:

We have below mentioned equation:

$$(pa^3 + qb^3 = pc^3 + qd^3)$$

For (p,q)=(1,86) above equation becomes

$$(1)a^3 + (86)b^3 = (1)c^3 + (86)d^3)$$

The above has numerical solution (a,b,c,d)=(9,2,11,1)

We substitute (a,b,c,d)=(9+t,2+kt,11+t,1+kt)

We also have (b-d) = [(2+kt)-(1+kt)] = 1 and

(c-a) = [(11+t)-(9+t)] = 2

After simplification we get

$$t = \left(\frac{20 - 129k}{43k^2 - 1}\right)$$

After substituting for t in (a, b, c, d) we get the below parameterization:

$$a = 387k^2 - 129k + 1$$

$$b = -43k^2 + 20k - 2$$

$$c = 473k^2 - 129k + 9$$

$$d = -86k^2 + 20k - 1$$

For k=1, we get numerical solution:

$$(1)(269)^3 + (86)(67)^3 = (1)(353)^3 + 86(25)^3$$

Degree (n=4)

We show solutions of quartic equation with coefficient 'h' for degree n=4

$a^4 + hb^4 = c^4 + hd^4$ --------------------- (3)

Gerardin gave a solution of $a^4 + hb^4 = c^4 + hd^4$ (Ref. no. 4)

$$(2k^2)^4 + h(k-1)^4 = (2k)^4 + h(k+1)^4$$

For $h = 2k^3(k^2 - 1)$

Let a = px+w, b = qx-1, c = px-w, d = qx+1, then (3) becomes (4) given below.

$(8wp^3 - 8hq^3)x^3 + (-8hq + 8w^3p)x$ .... 4)

To obtain a solution of (4), we have to find the rational solution of (5).

$$v^2 = -64w^4p^4 + 64whqp^3 + 64hq^3w^3p - 64h^2q^4 \ldots\ldots(5)$$

Let us consider eqn.(5) as a quadratic equation in h.

We find two solutions of (5) as follows.

h = $\left(\frac{\boldsymbol{w}}{\boldsymbol{q}}\right)(\boldsymbol{p})^3$

Substitute h = p^3(w/q) to (5), then

$v^2 = -64p^4w^2(p-w)(p+w)(q-1)(q+1)$.

Let: $w = m^2 + n^2, \quad p = m^2 - n^2, \quad q = \frac{m^2+n^2}{2mn},$
then $h = 2(m^2 - n^2)^3(mn)$

From eqn. (4) above we get: $x = \frac{4m^2n^2}{-2m^2n^2+n^4+m^4}$

We obtain a solution below: (Form A)
a = $(n-m)(n+m)(n^4 - 4m^2n^2 - m^4)$

b = $2m^3n + 2n^3m + 2m^2n^2 - n^4 - m^4$

c = $(n-m)(n+m)(n^4 + 4m^2n^2 - m^4)$

d = $2m^3n + 2n^3m - 2m^2n^2 + n^4 + m^4$

h = $2mn(m^2 - n^2)^3$

By using this new solution as a known solution, we obtain more new solutions.

Next we take h = $\frac{pw}{q}$, Similarly, (5) becomes to

$$v^2 = 64w^2p^2(w-1)(w+1)(p-q)(p+q)$$
$$v^2 = (8wp)^2(w^2-1)(p^2-q^2)$$

Let w = $\frac{m^2+n^2}{2mn}$ $q = m^2 + n^2, \quad p = m^2 - n^2$

Then $h = 8(m^2 - n^2)m^3n^3 \quad and \quad x = -\frac{1}{4}(\frac{-m^2+n^2}{m^2n^2})$

We obtain another solution below: (Form B)

$$a = 2mn(m^4 - 2m^2n^2 + n^4 + 2m^3n + 2mn^3)$$

$$b = m^4 - n^4 - 4m^2n^2$$

$$c = 2mn(m^4 - 2m^2n^2 + n^4 - 2m^3n - 2mn^3)$$

$$d = m^4 - n^4 + 4m^2n^2$$

$$h = 8m^3n^3(m^2 - n^2)$$

Alternate solution:
To get equation of the type below:

$$pa^4 + qb^4 = pc^4 + qd^4$$

We put n=1 in the equation [Form (B) above] and then substitute m=x/y, giving us the parametric form below.

$$a = 2x(x^4 - 2x^2y^2 + y^4 + 2x^3y + 2xy^3)$$

$$b = y(x^4 - y^4 - 4x^2y^2)$$

$$c = 2x(x^4 - 2x^2y^2 + y^4 - 2x^3y - 2xy^3)$$

$$d = y(x^4 - y^4 + 4x^2y^2)$$

$$p = y^5$$

$$q = 8x^3(x^2 - y^2)$$

For x=1 & y=2 we get the numerical solution:

$$(4)(29)^4 + 3(1)^4 = (4)(11)^4 + 3(31)^4$$

Alternate solution:

We have $(pa^4 + qb^4 = pc^4 + qd^4)$

We get:

$$\frac{p}{q} = \frac{d^4 - b^4}{a^4 - c^4}$$

Let a=u+v, b=v-w, c=u-v, d=v+w

By resolving above polynomial into factors, we get {p,q}.

$$p_4(u+v)^4 + q_4(v-w)^4 = p_4\,(u-v)^4 + q_4(v+w)^4$$

Where:

$$p_4 = w(v^2 + w^2)$$
$$q_4 = u(u^2 + v^2)$$

Let: (u,v,w)=(5,3,2) & we get: (p,q,a,b,c,d)=(26,170,8,1,2,5)

$$26(8)^4 + 170(1)^4 = 26(2)^4 + 170(5)^4$$

(See Table below for parametric solutions)

<u>Table (A) : </u> (degree fourth power)

$$(pa^4 + qb^4 = pc^4 + qd^4)$$

For p=f(m) & q=f(m,n), where f(n) & f(m,n) are a function in variable 'n' & '(m,n)' respectively

The following parametric solutions were found:

$(\boldsymbol{p},\boldsymbol{q}) =$ $[\boldsymbol{f}(\boldsymbol{m})$,**f(m,n)]**	$\boldsymbol{a}$ & $\boldsymbol{b}$	$\boldsymbol{c}$ & $\boldsymbol{d}$
$m^4, (n^2 + 4m^2)(n^2 + 2m^2)$	$n^2 - nm + 2m^2,$ $(n + m)m$	$n^2 + nm + 2m^2,$ $(n - m)m$
m^4 , $2(n^4 + 6n^2m^2 + m^4)$	$n^2 - 2nm + m^2,$ $(n + m)m$	$n^2 + 2nm + m^2,$ $(n - m)m$
$m^4, 2(n^2 + 9m^2) *$ $(n^2 + 3m^2)$	$n^2 - 2nm + 3m^2,$ $(n + m)m$	$n^2 + 2nm + 3m^2,$ $(n - m)m$
m^4, $3(n^2 + 4m^2) *$ $(n^2 - 2m^2)$	$n^2 - 3nm - 2m^2,$ $(n + m)m$	$n^2 + 3nm - 2m^2,$ $(n - m)m$
m^4, $3(n^4 + 11n^2m^2 + m^4)$	$n^2 - 3nm + m^2,$ $(n + m)m$	$n^2 + 3nm + m^2$, $(n - m)m$
$m^6, (n^6 + 2n^4m^2 + n^2m^4$ $+ m^6)$	$n^3 + nm^2 - m^3,$ $(n + m)m^2$	$n^3 + nm^2 + m^3,$ $(n - m)m^2$

Next for fifth power:

<u>Degree (n=5)</u>

We have eqn. $(pa^5 + qb^5 = pc^5 + qd^5)$

We get:

$$\frac{p}{q} = \frac{d^5 - b^5}{a^5 - c^5}$$

Let a=u+v, b=v-w, c=u-v, d=v+w
By resolving above polynomial into factors, we get {p,q}.

$$p_5(u+v)^5 + q_5(v-w)^5 = p_5\ (u-v)^5 + q_5(v+w)^5$$

Where:

$$p_5 \ = w(5v^4 + 10v^2w^2 + w^4)$$
$$q_5 = v(5u^4 + 10u^2v^2 + v^4)$$

Let: (u,v,w)=(5,3,2) & we get:

(p,q,a,b,,c,d)=(1562,16368,8,1,2,5)

$$1562(8)^5 + 16368(1)^5 \ = \ 1562(2)^5 + 16368(5)^5$$

<u>Degree (n=6)</u>

We have $(pa^6 + qb^6 = pc^6 + qd^6)$

We get:

$$\frac{p}{q} = \frac{d^6 - b^6}{a^6 - c^6}$$

Let a=u+v, b=v-w, c=u-v, d=v+w
By resolving above polynomial into factors, we get {p,q}.

$$p_6(u+v)^6 + q_6(v-w)^6 = p_6\ (u-v)^6 + q_6(v+w)^6$$

Where:

$$p_6 = w(v^2+3w^2)(3v^2+w^2)$$

$$q_6 = u(u^2+3v^2)(3u^2+v^2)$$

Let: (u,v,w)=(5,3,2)

(p,q,a,b,c,d)=(1302,21840,8,1,2,5)

$$1302(8)^6+21840(1)^6 = 1302(2)^6+21840(5)^6$$

Degree (n=7)

We have $(pa^7+qb^7=pc^7+qd^7)$

We get:

$$\frac{p}{q}=\frac{d^7-b^7}{a^7-c^7}$$

Let a=u+v, b=v-w, c=u-v, d=v+w

By resolving above polynomial into factors, we get {p,q}.

$$p_7(u+v)^7+q_7(v-w)^7=p_7(u-v)^7+q_7(v+w)^7$$

Where:

$$p_7 = [\mathrm{w}(7\mathrm{v}^6+35\mathrm{v}^4\mathrm{w}^2+21\mathrm{v}^2\mathrm{w}^4+\mathrm{w}^6)$$

$$q_7 = \mathrm{v}(7\mathrm{u}^6+35\mathrm{u}^4\mathrm{v}^2+21\mathrm{u}^2\mathrm{v}^4+\mathrm{v}^6)]$$

Let: (u,v,w)=(5,3,2) & we get for degree n=7 the below mentioned numerical solution:

(p,q,a,b,c,d)=(19531,524256,8,1,2,5)

$$19531(8)^7+524256(1)^7 = 19531(2)^7+524256(5)^7$$

Degree n=8

We have $(pa^8+qb^8=pc^8+qd^8)$

We get:

$$\frac{p}{q}=\frac{d^8-b^8}{a^8-c^8}$$

Let a=u+v, b=v-w, c=u-v, d=v+w
By resolving above polynomial into factors, we get {p,q}.

$$p_8(u+v)^8+q_8(v-w)^8=p_8\,(u-v)^8+q_8(v+w)^8$$

Where;

$$p_8=w(v^2+w^2)(w^4+6v^2w^2+v^4)$$
$$q_8=u(v^2+u^2)(v^4+6u^2v^2+u^4)$$

Let: (u,v,w)=(5,3,2) & we get numerical solution:

(p,q,a,b,c,d)=(4069,174760,8,1,2,5)

$$4069(8)^8+174760(1)^8\;=\;4069(2)^8+174760(5)^8$$

Degree n=9:

We have $(pa^9+qb^9=pc^9+qd^9)$

We get:

$$\frac{p}{q}=\frac{d^9-b^9}{a^9-c^9}$$

Let a=u+v, b=v-w, c=u-v, d=v+w
By resolving above polynomial into factors, we get {p,q}.

$$p_9(u+v)^9+q_9(v-w)^9=p_9\,(u-v)^9+q_9(v+w)^9$$

Where:

$$p_9=w(3v^2+w^2)(w^6+33v^2w^4+27v^4w^2+3v^6)$$

$$q_9=v(3u^2+v^2)(v^6+33u^2v^4+27u^4v^2+3u^6)$$

Let: (u,v,w)=(5,3,2)

(p,q,a,b,c,d)=(976562,266304,8,1,2,5)

$$976562(8)^9 + 266304(1)^9 = 976562(2)^9 + 266304(5)^9$$

On a side note, the interested reader may look at the below identities (A, B, C & D) that were derived for the equation (2) given below:

$$ma^n + nb^n = uc^n + vd^n \text{ --------(2)}$$

Here the coefficients for eqn.(2) are different than that used for equation (1) at the top of the page with the coefficients on the left hand side of equation (2) being different from the coefficients on the right hand side.

(m, n, u, v)=(1,3,1,4)

[A] $(1)(12x^8 + 6x^4 - 1)^4 + 3(2x)^4 = (1)(12x^8 - 6x^4 - 1)^4 + 4(12x^7)^4$

(m, n, u, v)=(1,6,1,2)

[B] $(1)(48x^8 + 12x^4 - 1)^4 + 6(2x)^4 = (1)(48x^8 - 12x^4 - 1)^4 + 2(48x^7)^4$

(m,n,u,v)=(1,24,1,8)

[C] $(1)(3x^8 + 3x^4 - 1)^4 + 24(x)^4 = (1)(3x^8 - 3x^4 - 1)^4 + 8(3x^7)^4$

(m,n,u,v)=(1,27,1,36)

[D] $(1)(972x^8 + 54x^4 - 1)^4 + 27(2x)^4 =$

$$(1)\ (972x^8 - 54x^4 - 1)^4 + 36(324x^7)^4$$

[See below for numerical solutions for equation (1) for different degree ‘n’]

Equation (1) ----- $(pa^n + qb^n = pc^n + qd^n) --$ for n=2,3,4,5,6,7,8 & 9

Degree (n)	p	q	a	b	c	d
2	2	3	11	1	1	9
3	13	62	5	1	1	3
4	5	3	5	4	1	6
5	25	4	34	9	12	49
6	19	5	4	2	1	5
7	547	32	6	1	2	9
8	353	60	4	3	2	5
9	729	19	4	3	1	6

(See below for list of references)

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