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Solution of Polynomial Equation of any Degree ’n’ with Special Emphasis for n= (2, 3, 4, 5 & 6)

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Absract

In this paper we deal with the below mentioned general polynomial equation of any degree ‘n’:

$$a_n x^n + a_{n-1} x^{n-1} y + - - - - + a_2 x^2 y^{n-2} + a_1 x y^{n-1} + a_0 y^n = a_n u^n + a_{n-1} u^{n-1} v + - - - - + a_2 u^2 v^{n-2} + a_1 u v^{n-1} + a_0 v^n \text{------------------ (A)}$$

First in section(I) we show various methods of solution for degree $n = (2,3,4,5,6)$ *& lastly in section (II) we show solution for degree seven after arriving at a general solution for the polynomial equation (A) for any degree 'n'.*

Keywords – (Symetrical Diophantine equations, Number Theory, Pure Math)

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1. Theory & Methods

Degree Two

$$a_2\,x^2 + a_1\,xy + a_0\,y^2 = a_2\,u^2 + a_1\,uv + a_0\,v^2$$

Put a0=2, a1=5, a2=3 we get

$$3x^2 + 5xy + 2y^2 = 3u^2 + 5uv + 2v^2$$

Let $x = t + n,\ y = -t + n,\ u = -t + m,\ v = t + m,$

This has parameterization in variables (m, n)

$$x = 5m - 4n, \quad y = -5m + 6n, \quad u = -4m + 5n, \quad v = 6m - 5n$$

Degree Three

$$a_3\,x^3 + a_2\,x^2y + a_1\,xy^2 + a_0\,y^3 = a_3\,u^3 + a_2\,u^2v + a_1\,uv^2 + a_0\,v^3$$

Put a_0=a_1=a_2=a_3=1 we get

$$x^3 + x^2y + xy^2 + y^3 = u^3 + u^2v + uv^2 + v^3$$

Let $u = t + n,\ v = t - n,\ x = pt + m,\ y = qt - m$

Substituting above & after parameterization we get,

$$x = m(2n^2m^2 - 2n^4 + m^4)$$

$$y = -m(2n^2m^2 + m^4 + 2n^4)$$

$$u = n(-2m^3n + m^4 + 2n^4)$$

$$v = -n(2m^3n + m^4 + 2n^4)$$

Degree Four

(See below section (II) after degree six)

Degree Five

$$a_5\,x^5 + a_4\,x^4y + a_3\,x^3y^2 + a_2\,x^2y^3 +$$

$$a_1\,xy^4 + a_0\,y^5 = a_5u^5 + a_4\,u^4v + a_3\,u^3v^2 + a_2\,u^2v^3 +$$

$$a_1\,uv^4 + a_0\,v^5$$

$$x^5 + 3x^4y + 3x^3y^3 + 2x^2y^3 + 2xy^4 + y^5 =$$

$$u^5 + 3u^4v + 3u^3v^3 + 2u^2v^3 + 2uv^4 + v^5$$

$$u = t + n, \quad v = t - n, \quad x = pt + m, \quad y = qt - m$$

Substituting above & parameterization we get

$$x = -m(37n^7w - 84n^2wm^5 + 47m^8)$$

$$y = m(-26n^7w - 21n^2wm^5 + 47m^8)$$

$$u = -n * (84m^3n^5 - 37m^8 - 47n^7w)$$

$$v = n(21m^3n^5 + 26m^8 - 47n^7w), \quad Where, \quad w^2 = mn$$

Degree Six

$$a_6\,x^6 + a_5\,x^5y + a_4\,x^4y^2 + a_3\,x^3y^3 + a_2\,x^2y^5 +$$

$$a_1\,xy^5 + a_0\,y^6 = a_6\,u^6 + a_5u^5v + a_4\,u^4v^2 + a_3\,u^3v^3 + a_2\,u^2v^4 +$$

$a_1\,uv^5 + a_0\,v^6$ ---------------------(1)

$$Let\ x = pt + 1, y = qt - 1, u = pt - 1, v = qt + 1 --- (2)$$
$$Substitute\ (2) to\ (1) and\ simplifying\ (1), we\ obtain$$

$$(8a4p^3q^2 + 2a1q^5 - 4a4p^4q + 4a2pq^4 - 6a3p^3q^2 - 2a5p^5 + 12a6p^5 - 12a0q^5 - 10a1pq^4 - 8a2p^2q^3 + 6a3p^2q^3 + 10a5p^4q)t^5$$
$$+(-40a0q^3 - 20a1pq^2 + 20a1q^3 - 8a2q^3 + 2a3q^3 - 20a5p^3 - 24a4p^2q + 20a5p^2q + 8a4pq^2 + 8a4p^3 + 18a3p^2q + 24a2pq^2 - 18a3pq^2 - 8a2p^2q + 40a6p^3 - 2a3p^3)t^3$$
$$+(4a2p - 4a4q - 8a2q - 2a1p + 6a3q - 12a0q + 10a1q + 2a5q + 8a4p - 10a5p + 12a6p - 6a3p)t \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

$$Equating\ to\ zero\ the\ coefficient\ of\ t\ in\ (3), then\ we\ obtain$$

$$a_5 = -3a3 + 2a4 + 4a2 + 6a0 - 5a1$$
$$a_6 = -2a3 + a4 + 3a2 + 5a0 - 4a1$$
$$And\ furthermore, \quad set$$
$$a_4 = 3a3 + 10a1 - 15a0 - 6a2.$$

$$To\ obtain\ the\ rational\ solution\ t\ of\ (3), discriminant\ must\ be\ square.$$
$$4(p+q)^6\big((-21a1 + 36a0 + 10a2 - 3a3)p^2 + (-2a2 - 18a0 + 8a1)q * p + (6a0 - a1)q^2\big)(a3 + 10a1 - 20a0 - 4a2) = square \ldots. (4)$$
$$Thus\ substitute\ the\ parametric\ solution\ of\ (4) to\ (2), then\ we\ obtain\ the$$

$$parametric\ solution\ of\ (1).$$
$$we\ obtain\ the\ parametric\ solution\ of\ degree\ one\ as\ follows.$$

$$Set\ a_3 = 4a2 - 10a1 + 20a0\ and\ a_4 = 6a2 - 20a1 + 45a0.$$
$$a_0, a_1, and\ a_2\ are\ arbitrary.$$

$$\begin{aligned} x &= (6a_0 - a_1)t + 1 \\ y &= (-9a_1 + 2a_2 + 24a_0)t - 1 \\ u &= (6a_0 - a_1)t - 1 \\ v &= (-9a_1 + 2a_2 + 24a_0)t + 1 \end{aligned}$$

Example

$$Let\ a_0 = 1, a1 = 2, a2 = 3, a3 = 12, a4 = 23, a5 = 18, a6 = 5$$

$$5x^6 + 18x^5y + 23x^4y^2 + 12x^3y^3 + 3x^2y^4 + 2xy^5 + y^6$$
$$= 5u^6 + 18u^5v + 23u^4v^2 + 12u^3v^3 + 3u^2v^4 + 2uv^5 + v^6$$
$$Solution\ is$$
$$x = 4t + 1, y = 12t - 1, u = 4t - 1, v = 12t + 1$$

For t = 2 we get (x , y, u, v) = (9, 23, 7, 25)

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2. Proof And Solving

$$We\ have\ the\ polynomial\ equation\ (A) given\ below:$$

$$a_nx^n + a_{n-1}x^{n-1}y + ----+a_2x^2y^{n-2} +$$
$$a_1xy^{n-1} + a_0y^n = a_nu^n + a_{n-1}u^{n-1}v + ----+$$
$$a_2u^2v^{n-2} + a_1uv^{n-1} + a_0v^n$$

Where { $a_{n,} a_{n-1}$, a_{n-2},-------a_4, a_3, a_2, a_1, a_0 } are coefficents of the above polynomial.

We fix the condition that (sum of even coefficients = sum of odd coefficients).

For quartic polynomial, we get after putting n=4 in eqn (A)

$$a_4\,x^4 + a_3\,x^3y + a_2\,x^2y^2 +$$
$$a_1\,xy^3 + a_0\,y^4 = a_4\,u^4 + a_3\,u^3v + a_2\,u^2v^2 + a_1\,uv^3 + a_0\,v^4$$

Example for coefficients of degree four we have

(a_4 + a2 + a0 = a3 + a_1)

Let x=pt+1, y=qt-1, u=pt-1, v=qt+1

We take ($a_{0,}a_{1,}a_2$, t) as parameters & for simplicity we call ($a_{0,}a_{1,}a_2$) = (d, e, f)

Thus we have (a_4 - a_3) = (e – d - f) and equation (A) above becomes for n = 4

After some algebra we get after solving for (p, q),

p = 4d - e , q = 8d - 5e + 2f, a_3 = (4d - 3e + f), a_4=(3d - 2e + f)

Similarly for Degree five

$$a_5\, x^5 + a_4\, x^4 y + a_3\, x^3 y^2 + a_2\, x^2 y^3 + a_1\, xy^4 + a_0\, y^5 = a_5 u^5 + a_4\, u^4 v + a_3\, u^3 v^2 + a_2\, u^2 v^3 + a_1\, uv^4 + a_0\, v^5$$

We get for (p, q)

p = 5d-e and q = 15d-7e+2f

Also we get, a_3 = (10d - 6e + 3f), a_4 = (15d - 8e + 3f), a_5 = (6d - 3e + f)

And so on for degrees (6, 7, 8, ------ etc).

The coefficients '' a_3" for degree (3, 4, 5, 6) are

a_3 = (d – e + f) for degree three

a_3 = (4d-3e+2f) for degree four

a_3 = (10d-6e+3f) for degree five

a_3 = (20d-10e+4f) for degree six

Thus the general form for the coefficient (a_3) = (r*d - s*e + w*f)

Putting the above in tabular form we have,

Coefficients (a_3)	Coefficient of d	Coefficient of e	Coefficient of f
Degree three	1	1	1
Degree four	4	3	2
Degree five	10	6	3
Degree six	20	10	4

The above three columns are in series, Hence they are represented as

$$Coefficent\ of\ `d'\ as, \quad r = (p * (p+2)(p+1))/6$$

$$Coefficent\ of\ \ `e'\ as, \quad s = \ (p(p+1))/2$$

$$Coefficent\ of\ `f'\ as, \quad w = \ (p)$$

Since the 1st term is p = 1 and the degree is n = 3,

(n, p) are related as p = (n-2). After putting value of (p, r, s, w) in

(a_3) = (r*d - s*e + w*f) we get

Thus we get coefficent $(a_3) = \frac{(n-2)}{n!} * (n(n-1)d - 3(n-1)e + 6f)$

In terms of degree 'n'

Thus (p,q) has representation in terms of degree 'n' as

P = (n*d -e) and q = [n(n-2)d - (2n-3)e + 2f]

And the coefficients (a3, a4, a5, ---) as

$a_3 = \frac{(n-2)}{n!} * (n(n-1)d - 3(n-1)e + 6f)$

$$a_4 = \frac{(n-2)(n-3)}{n!} * (3n(n-1)d - 8(n-1)e + 12f)$$

$$a_5 = \frac{(n-2)(n-3)(n-4)}{n!} *$$

$$(6n(n-1)d - 15(n-1)e + 20f)$$

& so on for other coefficients ($a_6, a_7, a_8,$ -----a_m,------a_n)

$$Where\ n!\ is\ `n'\ factorial, \quad n = (1 * 2 * 3 * 4 * ------ * n)$$

For integer m, value of coefficient (am) we arrive at,

$$a_m = \frac{(n-2)(n-3) - -(n-m+1)}{2 * (m!)} *$$

$$\big((m-1)(m-2)n(n-1)d - 2m(m-2)(n-1)e + 2m(m-1)f\big) ----(\mathrm{B})$$

For coeeficent a_6 put m=6 in the above & we get

$$a_6 = \frac{(n-2)(n-3)--(n-5)}{2*6!} *$$

$$(5*4n(n-1)d - 2*6(4)(n-1)e + 2*6*5f)$$

Which simplyfies as,

$$a_6 = \frac{(n-2)(n-3)--(n-5)}{n!}$$

$$*(10n(n-1)d - 24(n-1)e + 30f)$$

So now we solve by the new method, the polynomial equation of degree six by using the known coefficents above,

$$a_6\,x^6 + a_5\,x^5y + a_4\,x^4y^2 + a_3\,x^3y^3 + a_2\,x^2y^5 +$$

$$a_1\,xy^5 + a_0\,y^6 = a_6\,u^6 + a_5u^5v + a_4\,u^4v^2 + a_3\,u^3v^3 + a_2\,u^2v^4 +$$

$$a_1\,uv^5 + a_0\,v^6$$

Substituting the values of (m, n) in equation (B) for various coefficients

(a_3, a_4, a_5, a_6) we get,

For example for,

a_3 we put m=3, n=6

a_4 we put m=4, n=6

a_5 we put m=5, n=6

a_6 we put m=6, n=6

Thus we get after the above substitution

a_3= (20d-10e+4f),

a_4= (45d-20e+6f) and

a_5= (36d-15e+4f),

a_6= (10d-4e+f)

We know that p = (n*d - e) & q = [n(n-2)d - (2n-3)e + 2f]

Hence for n=6 we get p = (6d - e), (q = 24d - 9e + 2f)

We have, x = pt+1, y = qt-1, u = pt-1, v = qt+1

Let us take d=5 e=4 & f=3, t=2 & and

Since (a_0 = d, a_1=e & a_2= f, t = t)

Hence we have (a_0, a_1 , a_2 , t)=(5,4,3,2) and

p=26, q=90, a3=72, a4=163, a5=172, a6=37

Thus the sum of even & sum of odd coefficents are

(a_6+a4+a2+a_0) = (a_5+a3+a1)

(37+163+3+5) = (132+72+4) = 208

x=pt+1, y=qt-1, u=pt-1, v=qt+1

And we get (x, y, u, v) = (53,179,51,181)

After substituting the coefficients in the sixth degree polynomial we get the below mentioned equation,

$$a_6\,x^6 + a_5\,x^5y + a_4\,x^4y^2 + a_3\,x^3y^3 + a_2\,x^2y^5 +$$

$$a_1\,xy^5 + a_0\,y^6 = a_6\,u^6 + a_5u^5v + a_4\,u^4v^2 + a_3\,u^3v^3 + a_2\,u^2v^4 +$$

$$a_1\,uv^5 + a_0\,v^6$$

$$37x^6 + 132x^5y + 163x^4y^2 + 72x^3y^3 + 3x^2y^4 +$$

$$4xy^5 + 5y^6 = 37u^6 + 132u^5v + 163u^4v^2 + 72u^3v^3 +$$

$$3u^2v^4 + 4uv^5 + 5v^6$$

The later equation can easily be verified that

(x, y, u, v) = (53,179,51,181) is the solution to the sixth degree polynomial.

Degree, n=7

$$a_7\,x^7 + a_6\,x^6y + a_5\,x^5y^2 + a_4\,x^4y^3 + a_3\,x^3y^4 + a_2\,x^2y^4 +$$

$$a_1\,xy^6 + a_0\,y^7 = a_7\,u^7 + a_6\,u^6y + a_5\,uv^2 + a_4\,u^4v^3 + a_3\,u^3v^4 + a_2\,u^2v^4 +$$

$$a_1\,uv^6 + a_0\,v^7$$

For n=7 Using, a_0 = d, a_1 = e & a_2= f, t = t &

Substituting the values of (m, n) in equation (B) for various coeficents (a3, a4, a5, a6) we get,

For example for,

a_3 we put m=3, n=7

a_4 we put m=4, n=7

a_5 we put m=5, n=7

a_6 we put m=6, n=7

a_7 we put m=7, n=7.

Hence we get :

a_3= (35d-15e+5f), a_4 = (105d-40e+10f)

a_5= (126d-45e+10f), a_6 = (70d-24e+5f),

a_7=(15d-5e+f)

& we get p = (7d - e), (q = 35d - 11e + 2f)

p = 31, q=137, a_3 =130, a4=395 ,a5=480, a6=269 & a7=58

x=pt+1, y=qt-1, u=pt-1, v=qt+1

Hence for (ao, a1, a2, t) = (5,4,3,2) we get

p=31, q=137, a_3=130, a4=395, a5=480, a6=269, a_7=58

And (x, y, u, v)=(63,273,61,275) & we get the seventh degree polynomial given below

$$58\,x^7 + 269\,x^6y + 480\,x^5y^2 + 395\,x^4y^3 + 130\,x^3y^4 + 3\,x^2y^5 + 4\,x\,y^6 + 5\,y^7 =$$

$$58\,u^7 + 269\,u^6v + 480\,u^5v^2 + 395\,u^4v^3 + 130\,u^3v^4 + 3\,u^2v^5 + 4\,u\,v^6 + 5\,v^7$$

The later equation can easily be verified that (x, y, u, v) = (63, 273, 61, 275) is the solution to the seventh degree polynomial.

3. Conclusions

The later method can be used to solve polynomial equations of any degree 'n' with only four parameters (d, e, f, t). Also polynomial equations like the above can be generated which we know have guaranteed integer solutions.

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