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Generalized Parametric Solutions to Multiple Sums of Powers

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Abstract The work on equation (B) below has been studied by others for different degree 'n' and equation (A) below for degree five, has been previously published by Mr. Ajai Choudhry (Ref. no. 1). But combined systematic analysis for degrees 2,3,4,5,6,7,8 & 9 etc. has not been done before, as is done in this paper. C*onsider the below mentioned equations*: $as^n + bt^n + cu^n = aw^n + bx^n + cy^n - - - (A)$ and $as^n + bt^n + cu^n + dv^n = aw^n + bx^n + cy^n + dz^n - -(B)$

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1. Introduction

The equation (A) in this paper has three terms on either side in section one and equation (B) has four terms on both sides in section two. In other words this paper deals with degree's (2,3,4,5,6,7 & 8) for equation (B) & gives a generalized parametric solutions for any degree 'n' for equation (A). As is known that solving Diophantine equations for power five and above is quite difficult. In this paper we show that equation (A) given above is solvable when (s,t,u,w,x,y) are known and we provide methods to find integer coefficents (a,b,c) for which the above Diophantine equation is satisfied for degree<10. Similarly, analysis is provided for equation (B) also. This paper thus shows that parametric solutions are possible for the Diophantine equations (B) for degree n < 10 & gives a generalized parametric solution for equation (A) above for any degree 'n'**.**

2. Discussion (See Below)

1. Consider Equation

$$aS^n + bT^n + cU^n = aW^n + bX^n + cY^n \qquad (A)$$

Let S=pt+1, T=qt+1, U=t+1
and W=pt-1, X=qt-1, Y= t-1
& c = - (ap+bq)

After substituting in equation (A) and solving for the coefficients (a,b,c) we get:

For even degree’s (n = 4, 6 & 8)

n=4 we get :

$$a = -\left(2t^2(q^3-1)+2(q-1)\right)$$
$$b = (2t^2(p^3-1)+2(p-1))$$
$$c = -(2t^2(p^3-q^3)+2(p-q))$$

For n=6 we get:

$$a = -3t^4(q^5-1)-10t^2(q^3-1)-3(q-1)$$
$$b = 3t^4(p^5-1)+10\,t^2(p^3-1)+3(p-1)$$
$$c = -3t^4(p^5-q^5)-10t^2(p^3-q^3)-3(p-q)$$

For n = 8 we get:

$$a = -(4t^6(q^7-1)+28t^4(q^5-1)+28t^2(q^3-1)+4(q-1))$$
$$b = (4t^6(q^7-1)28t^4(p^5-1)+28\,t^2(p^3-1)+4(p-1))$$
$$c = -(4t^6(p^7-q^7)+28t^4(p^5-q^5)+28t^2(p^3-q^3)+4(p-q))$$

Note: Similarly for n=10, 12, 14, etc. similar pattern with the coefficient’s is given in the table below.

For odd degree’s (n = 3, 5, 7 & 9)

n=3 we get

$$a = -(3t^2q(q-1)-(q-1))$$
$$b = (3t^2p(p-1)-(p-1))$$
$$c = -(3t^2pq(p-q)-(p-q))$$

For n=5 we get,

$$a = -(5t^4q(q^3-1)+10t^2q(q-1)-(q-1))$$
$$b = (5t^4p(p^3-1)+10t^2p(p-1)-(p-1))$$
$$c = -(5t^4pq(p^3-q^3)+310pq(p-q)-(p-q))$$

For n=7 we get,

$$a = -(7t^6q(q^5-1)+35t^4q(q^3-1)+21t^2q(q-1)-(q-1))$$
$$b = (7t^6q(q^5-1)35t^4p(p^3-1)+21t^2p(p-1)-(p-1))$$
$$c = -(7t^6pq(p^5-q^5)+35t^4pq(p^3-q^3)+21t^2pq(p-q)-(p-q))$$

For n=9 we get,

$$a = -(9t^8q(q^7-1)+84t^6q(q^5-1)+126t^4q(q^3-1)+36t^2q(q-1)-(q-1))$$
$$b = (9t^8 * p(p^7-1)+84t^6 * p(p^5-1)+126t\text{^}4\,p(p^3-1)+36t〗\ \text{^}2p(p-1)-(p-1))$$
$$c = -(9t^8 * pq(p^7-q^7)+84t^6pq(p^5-q^5)+126t^4pq(p^3-q^3)+36t^2pq(p-q)-(p-q))$$

Note: Similarly for odd ‘n’ = 11,13,15, etc a similar pattern of the coefficient’s is given, in the attached tables.

Table 1. Pascal’s Triangle

n		a^*	b	c	d	e	f	g	h	j	k	L	m	n	p	q	r	s	t	u	
											1										
1										1		1									
2									1		2		1								
3								1		3		3		1							
4							1		4		6		4		1						
5						1		5		10		10		5		1					
6					1		6		15		20		15		6		1				
7				1		7		21		35		35		21		7		1			
8			1		8		28		56		70		56		28		8		1		
9		1		9		36		84		126		126		84		36		9		1	

We have the well known Pascal’s Triangle (given above)

Above is for different degree’s ‘n’:
Above is table of Coefficient’s of expansion of (m+1)^n
*Coefficient (a, b, c, d, e, f, --------u) for (m^p) and selected degree ‘n’ is given by,

$$C_p^n = \frac{n!}{[\,p! * (n-p)!\,]}$$

$$Where\ (n!)\ means\ ‘n’\ factorial$$

(See below for more tables)

Equation,

$$aS^n + bT^n + cU^n = aW^n + bX^n + cY^n \tag{A}$$

Table 2. For odd degree

Degree n	constant	Coeff. (t^2)	Coeff. (t^4)	Coeff. (t^6)	Coeff. (t^8)	Coeff. (t^{10})	Coeff. (t^{12})	Coeff. (t^{14})	Coeff. (t^{16})	Remarks
3	1	n	-	-	-	-				
5	1	2n	n	-	-	-				
7	1	3n	5n	n	-	-				
9	1	4n	14n	84	n	-				n=prime
11	1	5n	30n	42n	15n	n				
13	1	6n	55n	132n	99n	22n	n			

15	1	7n	91n	5005	429n	3003	455	n		n=prime
17	1	8n	140n	728n	1430n	1144n	364n	40n	n	

Where (a, b, c) are polynomials in (t)
For above table, coefficients of (t^2) is given by, =n(n-1)/2
For degree eleven we get coefficient of (t^2) =n*(11-1)/2= 5n
and the coefficients of (t^4) is given by =n (n-1)(n-2)(n-3)/24
For degree eleven we get coeff. of (t^4) = n*(11-1)*(11-2)*(11-3)/24=30n

Table 3. For degree 'n' even

Degree n	constant	Coeff. (t^2)	Coeff. (t^4)	Coeff. (t^6)	Coeff. (t^8)	Coeff. (t^10)	Coeff. (t^12)
4	2	2					
6	3	10	3				
8	4	28	28	4			
10	5	60	126	60	5		
12	6	110	396	396	110	6	
14	7	182	1001	1716	1001	182	7

The constant terms in above table has the relation given below Table (3),

K = (n)(n-1)(n-2)/12 for degree 12

For degree n=12, the coefficient of constant term is = (12*11*10)/12 = 110

Table 4. For odd degree 'n'

Coefficient's for polynomial (a) in equation (A) above	Representation for degree 'n' = (3,5,7,9, -------u)	Remarks
t^0	1	
t^2	n*(n-1)/(2!)	
t^4	n(n-1)(n-2)(n-3)/(4!)	
t^6	n(n-1)(n-2)(n-3)(n-4)(n-5)/(6!)	
t^p , (p=8)	n(n-1)(n-2)(n-3)(n-4)(n-5)(n-6)(n-p+1)/(p!)	
-- -	----------------	
---	------------------	
---	----------------	
t^p	$= \frac{n!}{[\,(p!)(n-p)!\,]}$	Same formula as Pascal's table C_p^n

Where, (p!), represents (p) factorial

Table 5. For even degree 'n'

Coeeficent's for polynomial (a) in equation (A) above	Representation for degree 'n' = (4,6,8,10, -------v)	Remarks
t^0	2n(n-1)(n-2)/(4!)	
t^2	3n(n-1)(n-2)(n-3)(n-4)/(6!)	
t^4	4n(n-1)(n-2)(n-3)(n-4)(n-5)(n-6)/(8!)	
t^6	5n(n-1)(n-2)(n-3)(n-4)(n-5)(n-6)(n-7)(n-8)/(10!)	
t^p (p=8)	[(p+4)(n)(n-1)(n-2)(n-3)(n-4)(n-5)(n-6)(n – p - 2)/2] * ((p+4)!)	
-- -	----------------	
---	---------------	
---	----------------	
t^p	$= \frac{n!}{[\,(w)!\,(n-w)!\,]}$	This representation is similar like the form given in table (a) above. C_w^n

Where, (w!) represents [(p+1)!] factorial & 'n' is relevant degree of the equation.

For n=9 we get the following equations for the coefficients (a,b,c) of the equation (A) and is given below,

$$a \;=\; -(\,C_8^n * \,t^8 q(q^7-1) + C_{6*}^n t^6 q(q^5-1) + C_{4*}^n t^4 q(q^3-1) + C_{2*}^n t^2 q(q-1) - C_{0*}^n (q-1))$$

$$b \;=\; (C_{8*}^n t^8 * \,p(p^7-1) + C_6^n * t^6 * p(p^5-1) + C_4^n * t^4 p(p^3-1) + C_{2*}^n t\text{^}2p(p-1) - C_{0*}^n (p-1))$$

$$c = -(C_{8*}^n t^8 * \,pq(p^7-q^7) + C_{6*}^n t^6 pq(p^5-q^5) + C_4^n * t^4 pq(p^3-q^3) + C_{2*}^n t^2 pq(p-q) - C_{0*}^n (p-q))$$

Putting n=9 we get,

$$a \;=\; -(\,C_8^n * \,t^8 q(q^7-1) + C_{6*}^n t^6 q(q^5-1) + C_{4*}^n t^4 q(q^3-1) + C_{2*}^n t^2 q(q-1) - C_{0*}^n (q-1))$$

$$= \;-(9t^8 q(q^7-1) + 84t^6 q(q^5-1) + 126t^4 q(q^3-1) + 36t^2 q(q-1) - (q-1))$$

Similarly we get:

$$b \;=\; (9t^8 * \,p(p^7-1) + 84t^6 * p(p^5-1) + 126t\text{^}4\,p(p^3-1) + 36t^2 p(p-1) - (p-1))$$

$$c = -(9t^8 * \,pq(p^7-q^7) + 84t^6 pq(p^5-q^5) + 126t^4 pq(p^3-q^3) + 36t^2 pq(p-q) - (p-q))$$

General equation for degree 'n' odd is given below,

$$a \;=\; -(\,C_{(n-1)}^n * \,t^{(n-1)} q\big(q^{(n-2)}-1\big) + C_{(n-3)*}^n t^{(n-3)} q\big(q^{(n-4)}-1\big) + C_{(n-5)*}^n t^{(n-5)} q\big(q^{(n-6)}-1\big) + -------- ----- + C_{2*}^n t^2 q(q-1) - C_{0*}^n (q-1))$$

And similarly for (b & c)

For even, n = 8 we get:

$$a \;= \left(-\tfrac{1}{2}\right) * (C_w^n * t^6 (q^7-1) + C_w^n * t^4 (q^5-1) + ----- +$$

$$C_w^n * t^2 (q^3-1) + C_w^n * (q-1))$$

$$b \;= \left(-\frac{1}{2}\right) * (8t^6 (q^7-1) + 56t^4 (p^5-1) + 56\;t^2 (p^3-1) + 8(p-1))$$

$$c = \left(-\frac{1}{2}\right) * (8t^6 (p^7-q^7) + 56t^4 (p^5-q^5) + 56t^2 (p^3-q^3) + 8(p-q))$$

Where (w=p+1)

Hence general solution for even degree 'n' is given below,

$$a = \left(-\frac{1}{2}\right) * (C^n_{(n-1)} * t^{(n-2)}\left(q^{(n-1)} - 1\right) + C^n_{(n-3)} * t^{(n-4)}\left(q^{(n-3)} - 1\right) + ----------$$

$$+ C^n_3 * t^2(q^3 - 1) + C^n_1 * (q - 1))$$

For example in the above for n=8 & coefficient of $(t\hat{}p) = (t\hat{}2)$

we get for p=2. w=p+1=2+1=3 ,

$$C^n_w = C^8_3 = \frac{(8*7*6*5*4*3*2*1)}{(5*4*3*2*1)*(3*2*1)} = 56$$

$$a = \left(-\frac{1}{2}\right) * (C^n_7 * t^6(q^7 - 1) + C^n_5 * t^4(q^5 - 1) +$$

$$C^n_5 * t^2(q^3 - 1) + C^n_1 * (q - 1))$$

$$= \left(-\frac{1}{2}\right) * (8 * t^6(q^7 - 1) + 56 * t^4(q^5 - 1) +$$

$$56 * t^2(q^3 - 1) + 8 * (q - 1))$$

$$= -(4 * t^6(q^7 - 1) + 28 * t^4(q^5 - 1) + 28 * t^2(q^3 - 1) + 4 * (q - 1))$$

Similarly we get,

$$b = (4t^6(q^7 - 1)28t^4(p^5 - 1) + 28\,t^2(p^3 - 1) + 4(p - 1))$$

$$c = -(4t^6(p^7 - q^7) + 28t^4(p^5 - q^5) + 28t^2(p^3 - q^3) + 4(p - q))$$

Numerical solutions by taking values of variable (p, q, t) = (3,2,2) is given in the below table

$$(s, t, u, v) = (pt + 1, qt + 1, rt + 1, t + 1)$$

$$(w, x, y, z) = (pt - 1, qt - 1, rt - 1, t - 1)$$

Table 6. Below for odd degree 'n' find numerical solutions

Degree 'n'	a	b	c	s	t	u	w	x	y
3	-23	70	-71	7	5	3	5	3	1
5	-1199	6478	-9359	7	5	3	5	3	1
7	-35783	369430	-631511	7	5	3	5	3	1
9	-947039	19170718	-35500319	7	5	3	5	3	1

Section (2)

$$as^n + bt^n + cu^n + dv^n = aw^n + bx^n + cy^n + dz^n \quad (B)$$

$$Let\ (s, t, u, v) = (pm + 1, qm + 1, rm + 1, m + 1) \quad (C)$$

$$(w, x, y, z) = (pm - 1, qm - 1, rm - 1, m - 1) \quad (D)$$

***For degree two*:**

From equation (B) for n = 2 we have,

$$as^2 + bt^2 + cu^2 + dv^2 = aw^2 + bx^2 + cy^2 + dz^2$$

Substituting & simplifying we get,

$$a(s^2 - w^2) + b(t^2 - x^2) + c(u^2 - y^2) + d(v^2 - z^2) = 0$$

and we get the condition, $ap + bq + cr + d = 0$

The above has solution

$$(a, b, c, d) = (1,1,1,-9)\ \&\ (p, q, r) = (4,3,2)$$

For $m = 2$ *in equation (C) & (D) we get*

$$(s, t, u, v) = (9,7,5,3)$$

$$(w, x, y, z) = (7,5,3,1)$$

Hence we get:

$$1(9)^2 + 1(7)^2 + 1(5)^2 + (-9)(3)^2 =$$
$$1(7)^2 + 1(5)^2 + 1(3)^2 + (-9)(1)^2$$

Degree Three :

$$as^3 + bt^3 + cu^3 + dv^3 = aw^3 + bx^3 + cy^3 + dz^3 - -(\mathrm{B})$$
$$Let\ (s, t, u, v) = (pm + 1, qm + 1, rm + 1, m + 1) - - - (\mathrm{C})$$
$$(w, x, y, z) = (pm - 1, qm - 1, rm - 1, m - 1) - - - (\mathrm{D})$$

Substituting & simplifying we get :

$$a(s^3 - w^3) + b(t^3 - x^3) + c(u^3 - y^3) + d(v^3 - z^3) = 0$$
$$(6d + 6ap^2 + 6bq^2 + 6cr^2)m^2 + 2a + 2b + 2c + 2d = 0$$

Hence we have to find rational number

$\{a, b, c, d, p, q, r, h\}$.

$$h^2 = -12p^2a^2 + \left((-12q^2 - 12p^2)b - 12d - 12cr^2 - 12p^2(c + d)\right)a$$
$$-12q^2b^2 + \left(-12d - 12cr^2 - 12q^2(c + d)\right)b - 12(d + cr^2)(c + d)$$

Let use a known solution $[p, q, r] = [4, 3, 2]$,

$$[a, b, c, d] = [1, 2, 6, -10], h = [24], m = 1/12$$

Substitute

the above to (C) & (D), *then we get a parametric*

solution as follows.

$$[a, b, c, d] = [k\char`^2 - 48k - 504,$$
$$2k^2 - 48k + 936, 6k^2 - 48k + 6696, -10k^2 - 48k - 16344]$$
$$[s, t, u, v] = [8k - 564, 9k - 468, 10k - 372, 11 * k - 276]$$
$$[w, x, y, z] = [-16k - 204, -15k - 108, -14k - 12, -13k + 84]$$

For $k = 0$ *we get*:

$$(a, b, c, d) = (7, -13, -93, 227)$$
$$(s, t, u, v) = (47,39,31,23)$$
$$(w, x, y, z) = (17,9,1, -7)$$
$$7(47)^3 + (-13)(39)^3 + (-93)(31)^3 + (227)(23)^3 =$$
$$7(17)^3 + (-13)(9)^3 + (-93)(1)^3 + (227)(-7)^3$$

Degree Four:

$$as^4 + bt^4 + cu^4 + dv^4 = aw^4 + bx^4 + cy^4 + dz^4$$
$$Let\ (s, t, u, v) = (pm + 1, qm + 1, rm + 1, m + 1)$$
$$(w, x, y, z) = (pm - 1, qm - 1, rm - 1, m - 1)$$

Substituting & simplifying we get:

$$a(s^4 - w^4) + b(t^4 - x^4) + c(u^4 - y^4) + d(v^4 - z^4) = 0$$
$$(8d + 8ap^3 + 8bq^3 + 8cr^3)m^3 + (8d + 8ap + 8bq + 8cr)m = 0$$

We get from above,

$$c = -\frac{ap^3 + bq^3 - ap - bq}{r(-1 + r^2)}$$

$$d = \frac{-r^2ap + ap^3 + bq^3 - r^2bq}{(-1 + r^2)}$$

$$Next, find\ rational\ number\ \{a, b, c, d, p, q, r\}.$$

$$Let\ us\ use\ a\ known\ solution\ [p, q, r] = [4, 3, 2],$$

$Parametric\ solution$:

$$(a, b, c, d) = \begin{pmatrix} k^2 - 576k + 133120, k^2 - 576k + 133120, \\ k^2 - 576k + 133120, -27k^2 - 576k - 1658880 \end{pmatrix}$$

$$(s, t, u, v) = (2k + 1440, k + 1280, 1120, k - 960)$$

$$(w, x, y, z) = (-6k + 160, -5k + 320, -4k + 480, -3k + 640)$$

Numerical solution is:

$$(a, b, c, d) = (1, 1, -14, 21)$$

$$(s, t, u, v) = (9, 7, 5, 3)$$

$$(w, x, y, z) = (7, 5, 3, 1)$$

Degree Five

$$as^5 + bt^5 + cu^5 + dv^5 = aw^5 + bx^5 + cy^5 + dz^5$$

$$Let\ (s, t, u, v) = (pm + 1, qm + 1, rm + 1, m + 1)$$

$$(w, x, y, z) = (pm - 1, qm - 1, rm - 1, m - 1)$$

$$Substituting\ \&\ simplifying\ we\ get:$$

$$a(s^5 - w^5) + b(t^5 - x^5) + c(u^5 - y^5) + d(v^5 - z^5) = 0$$

$$(10ap^4 + 10bq^4 + 10cr^4 + 10d)m^4$$

$$+(20ap^2 + 20bq^2 + 20cr^2 + 20d)m^2 + 2a + 2b + 2c + 2d = 0$$

$$First, substitute\ d = -a - b - c\ to\ above\ equation.$$

$$Next, find\ rational\ number\ \{a, b, c, d, p, q, r, h\}.$$

$$Let\ use\ a\ known\ solution\ [p, q, r] = [4, 3, 2],$$

$$[a, b, c, d] = [1, 1, -9, 7]$$

$$h^2 = -200(-1 + p^4)(-1 + p^2)a^2 +$$

$$((-200(-1 + q^4)(-1 + p^2) - 200(-1 + p^4)(-1 + q^2))b$$

$$-200(cr^4 - c)(-1 + p^2) - 200(-1 + p^4)(cr^2 - c))a$$

$$-200(-1 + q^4)(-1 + q^2)b^2 + (-200(cr^4 - c)(-1 + q^2)$$

$$-200(-1 + q^4)(cr^2 - c))b - 200(cr^4 - c)(cr^2 - c)$$

$$V = 400$$

$$[a, b, c, d] = [k^2 - 800k + 1060000,$$

$$k^2 - 800k + 1060000, -9k^2 - 800k - 17140000,$$

$$7k^2 + 2400k + 15020000]$$

$$[s, t, u, v] = [k - 13900, 2k - 11300, 3k - 8700, 4k - 6100]$$

$$[w, x, y, z] = [-9k - 6900, -8k - 4300, -7k - 1700, -6k + 900]$$

Degree six:

$$as^6 + bt^6 + cu^6 + dv^6 = aw^6 + bx^6 + cy^6 + dz^6$$

$$Let\ (s, t, u, v) = (pm + 1, qm + 1, rm + 1, m + 1)$$

$$(w, x, y, z) = (pm - 1, qm - 1, rm - 1, m - 1)$$

$$Substituting\ \&\ simplifying\ we\ get:$$

$$a(s^6-w^6)+b(t^6-x^6)+c(u^6-y^6)+d(v^6-z^6)=0$$
$$(12ap^5+12d+12cr^5+12bq^5)m^5$$
$$+(40bq^3+40ap^3+40d+40cr^3)m^3+(12cr+12bq+12ap+12d)m=0$$
$$First, substitute\ d=-cr-bq-ap\ to\ above\ equation.$$
$$Next, find\ rational\ number\ \{a,b,c,d,p,q,r\}$$
$$Let\ use\ a\ known\ solution\ [p,q,r]=[4,3,2],$$
$$[a,b,c,d]=[1,1,-18,29]$$

$\boldsymbol{Parametric\ Solution}$:

$$(a,b,c,d)=(k^2-5760k+39484800,$$
$$k^2-5760k+39484800,-18k^2-5760k-1019347200,$$
$$29k^2+51840k+1762300800)$$
$$(s,t,u,v)=([k+58680,47880,k-37080,2k-26280])$$
$$(w,x,y,z)=([7k+27720,6k+16920,5k+6120,4k-4680])$$

$\boldsymbol{Degree\ Seven}$:

$$as^7+bt^7+cu^7+dv^7=aw^7+bx^7+cy^7+dz^7$$
$$Let\ (s,t,u,v)=(pm+1,qm+1,rm+1,m+1)$$
$$(w,x,y,z)=(pm-1,qm-1,rm-1,m-1)$$
$$Substituting\ \&\ simplifying\ we\ get:$$
$$a(s^7-w^7)+b(t^7-x^7)+c(u^7-y^7)+d(v^7-z^7)=0$$
$$(14ap^6+14d+14cr^6+14bq^6)m^6$$
$$+(70bq^4+70ap^4+70d+70cr^4)m^4$$
$$+(42cr^2+42bq^2+42ap^2+42d)m^2$$
$$+2(a+b+c+d)=0$$
$$We\ take\ (c,d)\ as\ shown\ below, we\ also\ have\ d=-(a+b+c)$$
$$c=-\frac{bq^2+ap^2-a-b}{r^2-1}$$
$$d=\frac{-r^2a-r^2b+bq^2+ap^2}{r^2-1}$$
$$Hence\ we\ have\ to\ find\ rational\ number\ \{a,b,c,d,p,q,r\}.$$
$$Example: [p,q,r]=[5,4,2],[a,b,c,d]$$
$$=[11,-35,87,-63]$$
$$h^2=-980(r^2-r^2p^2+p^4-p^2)(r^4-r^4p^2+r^2-r^2p^2+p^6-p^2)a^2$$
$$+(-980(r^2-r^2q^2+q^4-q^2)(r^4-r^4p^2+r^2-r^2p^2+p^6-p^2)$$
$$-980(r^2-r^2p^2+p^4-p^2)(r^4-r^4q^2+r^2-r^2q^2+q^6-q^2))ba$$
$$-980(r^2-r^2q^2+q^4-q^2)(r^4-r^4q^2+r^2-r^2q^2+q^6-q^2)b^2$$
$$Thus\ it\ is\ sufficient\ to\ solve\ the\ quadratic$$
$$equation\ for\ \{a,b,h\}.$$
$$We\ get\ a\ parametric\ solution$$
$$a=11k^2-317520k+130557873600$$
$$b=-35k^2-317520k-452218334400$$

$$c = 87k^2 + 4127760k + 1216628683200$$
$$d = -63k^2 - 3492720k - 894968222400$$

$$[s,t,u,v] = [-2k - 982800, -k - 839160, k - 551880, 2k - 408240]$$
$$[w,x,y,z] = [-8k - 453600, -7k - 309960,$$
$$-5k - 22680, -4k + 120960]$$
$$k\ is\ arbitrary.$$

$$\boldsymbol{Degree\ Eight}:$$
$$as^8 + bt^8 + cu^8 + dv^8 = aw^8 + bx^8 + cy^8 + dz^8$$
$$Let\ (s,t,u,v) = (pm+1, qm+1, rm+1, m+1)$$
$$(w,x,y,z) = (pm-1, qm-1, rm-1, m-1)$$
$$Substituting\ \&\ simplifying\ we\ get:$$
$$a(s^8 - w^8) + b(t^8 - x^8) + c(u^8 - y^8) + d(v^8 - z^8) = 0$$
$$In\ a\ similar\ way:$$
$$we\ have\ to\ find\ rational\ number\ \{a,b,c,d,p,q,r\}.$$
$$We\ have\ known\ solution: [p,q,r] = [4,3,2],$$
$$[a,b,c,d] = [1,-7,18,-19]$$
$$Parametric\ solution\ is:$$
$$a = k^2 - 53760k + 23843635200$$
$$b = -7k^2 - 53760k - 178465996800$$
$$c = 18k^2 + 752640k + 475427635200$$
$$d = -19k^2 - 1128960k - 510831820800$$
$$[s,t,u,v] = [-2k - 1021440, -k - 833280, -645120, k - 456960]$$
$$[w,x,y,z] = [-6n - 483840, -5n - 295680, -4n - 107520, -3n + 80640]$$

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