$Parametric\ Solution\ to\ \ p(a)^n + q(b)^n = r(c)^n\ for\ degree\ \ \ n = 2,3,4,5\ \&\ 6$

ABSTRACT

Historically equation $(pa^n + qb^n = rc^n)$ has been studied for degree 2 and equation $(pa^n + qb^n = rc^n)$ herein called equation (1) has been studied for n=2, p=1, q=9 (Ref.no. 4) by Ajai Choudhry. Tito Piezas has discussed about equation (1) when p=r=1 (Ref. no. 3). While Ref. no. (3 & 4) deals with equation no. (1) for degree n=2, this paper has provided parametric solutions for degree n=2,3,4,5 & 6. Also there are instances in this paper where parametric solutions have been arrived at using different methods.

Keywords: Diophantine equations, Equal sums of powers, pure mathematics.

We begin with equation (1) for degree n=2 & than go to n=3,4,5 & 6

Degree n=2:

$$p(a)^2 + q(b)^2 = r(c)^2$$

We have known solution:

$$35(a)^2 + 10(b)^2 = 3(c)^2 \quad -----(1)$$

For (a,b,c)=(1,2,5)
Set (a,b,c)=(t+1),(t+2),(t*m+5)
in equation (1) & after simplification we get:

$$t = \frac{10(11-3m)}{3(m^2-15)}$$

Substituting value of 'm' for (a,b,c))=((t+1),(t+2),(t*m+5)) &
we get parametric solution of (a,b,c)

$$(a,b,c) = \big((3m^2+30m-155),(6m^2+30m-200),(45m^2-110m-225)\big)$$

Since, (p,q,r)=(35,10,3)
For m=2, we get:

$$35(19)^2 + 10(2)^2 = 3(65)^2$$

Another solution:

$$(a,b,c) = \big((2w-4),(w+1),(3w-3)\big)$$

$$(p,q,r) = \big((2w^2-2w+2),(w^2+2w-5),(w^2-2w+3)\big)$$

For w=4 we get:

$$(a,b,c) = (4{,}5{,}9)\ \&\ (p,q,r) = (26{,}19{,}11)$$

Degree n=3

$$p(a)^3 + q(b)^3 = r(c)^3 - - - - - (1)$$

Let (a,b,c)=[m-1,m,m+1]
and (p,q,r)=((w+1)(w)(w-1))
Substituting in (1) we get after simplification,

$$w = \frac{2m(m^2+3)}{(-m^3+6m^2+2)}$$

Substituting value of 'w' in (p,q,r)=((w+1)(w)(w-1) we get solution,

$$(p,q,r) = [\,(m^3+6m^2+6m+2), (2m^3+6m), (3m^3-6m^2+6m-2)\,]$$

For m=6 & removing common factor's we get:

$$235(5)^3 + 234(6)^3 = 233(7)^3$$

Degree n=4,

$$p(a)^4 + q(b)^4 = r(c)^4 - - - - - (1)$$

Let, $r = (a^2 + 2b^2)$

and $c^2 = b^2 - a^2$

Substitute values of 'c' & 'r' & after re-arranging terms we get:

We get:

$$a^4(p-r) + b^4(q-r) = a^4(-2b^2) + b^4(-4a^2)$$

$$Equating\ coefficent's\ of\ (a^4\ \&\ b^4)\ we\ get:$$

$$(p-r) = (-2b^2)\ and\ (q-r) = (-4a^2)$$

Hence for, $r = (a^2 + 2b^2)$ we get:

$$p = (a^2), \quad q = (2b^2 - 3a^2)\ and\ r = (a^2 + 2b^2)$$

We have numerical solution: $3^2 = 5^2 - 4^2$

hence (a,b,c)= (4,5,3)

and (p,q,r)=(16,2,66)

Hence we have,

$$16(4)^4 + 2(5)^4 = 66(3)^4$$

Degree n=5,

$$p(a)^5 + q(b)^5 = r(c)^5 \quad --------------------(1)$$

(a,b,c)=((m-1),(m),(m+1)) ----------------(2)
(p,q,r)=((w+1),(w),(w-1)) ------------------(3)

Substitute in equation (1) & Solve for w & substitute For (a,b,c,p,q,r) in (2) & (3) we get

$$w = \frac{2m(m^4 + 10m^2 + 5)}{(-m^5 + 10m^4 + 20m^2 + 2)}$$

substituting 'w' in (p,q,r)=((w+1),(w),(w-1)) we get,

$$p = (m^5 + 10m^4 + 20m^3 + 20m^2 + 10m + 2)$$
$$q = 2m(m^4 + 10m^2 + 5)$$
$$r = (3m^5 - 10m^4 + 20m^3 - 20m^2 + 10m - 2)$$

For m=3,
hence (a,bc)=(2,3,4)
(p,q,r)=(1805,1056,307)

$$1805(2)^5 + 1056(3)^5 = 307(4)^5$$

Degree n=6,

$$p(a)^6 + q(b)^6 = r(c)^6 ----- (1)$$

Let , $c^2 = (a^2 + b^2)$

and let $r = (a^4 + b^4)$
Substituting value of 'c' & 'r' in equation (1) and re-arranging the terms we get:

$$a^6(p-r) + b^6(q-r) = a^6\left(3b^2(a^2+b^2)\right) + b^6\left(3a^2(a^2+b^2)\right)$$

$$Equating\ coefficent's\ of\ (a^6\ \&\ b^6) we\ get:$$
$$(p-r) = 3b^2(a^2+b^2)$$
$$(q-r) = 3a^2(a^2+b^2)$$

$$Since \quad r = (a^4+b^4)\ we\ get:$$

$$p = a^4 + 3a^2b^2 + 4b^4$$
$$q = 4a^4 + 3a^2b^2 + b^4$$
$$r = a^4 + b^4$$

We have numerical solution $5^2 = 3^2 + 4^2$

Hence (a,b,c)=(3,4,5) & (p,q,r)= (1537,1012,337)

$$1537(3)^6 + 1012(4)^6 = 337(5)^6$$

Table for degree n= 2,3,4,5 & 6

Degree 'n'	p	q	r	a	b	c
2	7	2	25	19	2	13
3	235	234	233	5	6	7
4	125	29	125	3	10	7
5	227	122	1	2	5	3
6	1012	1537	337	3	4	5

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