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Taxicab Equations for Power Two, Three, Four & Five

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Abstract

This paper is primarily concerned with the below mentioned equation.

Equation (A):

$$2(w)^k = a^k + b^k + c^k = d^k + e^k + f^k = g^k + h^k + j^k = l^k + m^k + n^k$$

For k=2,3,4 the above equation has three terms in each of the four chains. For k=5 , the right hand side of the equation has six terms in each of the four chains.

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Parameterization of a chain of equations, namely,

$$(a^k + b^k + c^k) = (d^k + e^k + f^k),$$

which is of two chain lengths has been done by several authors. But parameterization of four chain lengths are not common. This paper has made inroads on this topic in that sense that and as an added bonus equated and tied these chain of equation to twice a kth power. Till now the sum of taxicab powers in math literature have summed up to an arbitrary integer. So the achievement of this paper is that by fixing the sum to twice a kth power, this equation (A) can be clubbed into the subject heading of equal sums of like powers. Now instead of having the sum equal to one kth power as in many cases in math literature we now have the sums equal to twice a kth power. In this paper we will designate this taxicab chain as k-2-3-3-3-3 where k=2, 3, 4 (degree). Except for 5 th powers the equation will be k-2-6-6-6 for reasons we will describe later in the paper.

In connection to our equation (A),there is an interesting anecdote about the conversation Ramanujan & Hardy had about a hundred years ago when Hardy came in a cab to visit Ramanujan. Hardy remarked that his cab's license plate was an uninteresting number 1729. But Ramanujan who had the ability of having integers as his friends said on the contrary 1729 is the smallest number which could be summed as two cubes in two different ways. Meaning sum of two cubes in two ways that is, (1729) = $1^3 + 12^3 = 9^3 + 10^3$. Since than mathematicians have called this type of equation as taxicab equation without the restriction on the powers being quadratic, cubic, quartic or quintic.

$$Nowadays\ a\ grade\ six\ student\ if\ asked\ to\ express\ 50\ as\ a$$

$$sum\ of\ two\ squares, he\ or\ she\ \ might\ say\ \ 50 = 1^2 + 7\text{^}2$$

$$But\ actually\ 50\ can\ be\ summed\ up\ as\ two\ squares\ in\ two\ ways\ as$$

$$50\ = 1^2 + 7^2 = 5^2 + 5^2$$
$$= 2(5)^2.\ The\ higher\ we\ go\ on\ the\ number\ line\ the\ more\ chains\ we\ can\ add.$$

$$For\ example,$$

$$2(w)^2 = 2(65)^2 =\ (13{,}91)^2 = (89{,}23)^2 = (85{,}35)^2 = (79{,}47)^2$$

$$This\ above\ chain, having\ four\ pair\ of\ square\ numbers, after\ paramatazation\ ,$$

$$we\ get\ a\ new\ numerical\ solution\ (k - 2 - 2 - 2 - 2 - 2)$$

$$where\ k = 2.$$

2nd degree taxi cab 2-2-2-2-2-2

The parametric form for the above is given by :

$$2*m^2(x^2+y^2)^2$$

$$= (-ay^2+ax^2-2bxy)^2+ (-by^2+bx^2+2axy)^2$$

$$=(-cy^2+cx^2-2dxy)^2+ (-dy^2+dx^2+2cxy)^2$$

$$= (-ey^2+ex^2-2fxy)^2+ (-fy^2+fx^2+2exy)^2$$

$$=(-gy^2+gx^2-2hxy)^2+ (-hy^2+hx^2+2gxy)^2$$

$Where\ 2(m)^2 = a^2+b^2 = c^2+d^2 = e^2+f^2 = g^2+h^2$

This has solution:

(a,b,c,d,e,f,g,h,m) = (13,91,89,23,85,35,79,47,65)

$$2*(65)^2 = (13,91)^2 = (89,23)^2 = (85,35)^2 = (79,47)^2$$

Consider the below mentioned equation, (1st chain)

$$2*m^2(x^2+y^2)^2$$

$$= (-ay^2+ax^2-2bxy)^2+ (-by^2+bx^2+2axy)^2$$

$$Where\ 2(m)^2 = a^2+b^2 =$$

$$Let\ k = \frac{x}{y}$$

$$2*m^2(k^2+1)^2 = (a^2+b^2)*(k^2+1)^2$$

$$= (-a+ak^2-2bk)^2+ (-b+bk^2+2ak)^2$$

$The\ right\ hand\ side\ after\ simplification\ becomes$

$=(a^2(k^2-1)^2-4ab(k^3-1)+4b^2k^2)+(b^2(k^2-1)^2+4ab(k^3-1)+$
$4a^2k^2)$

$=(a^2+b^2)(k^2-1)^2+4k^2(a^2+b^2)$

$=(a^2+b^2)*(k^2+1)^2 \quad (which\ equals\ the\ left\ hand\ side).$

Since $(a^2+b^2)=\ (c^2+d^2)$
$By\ substituting\ a=c\ \&\ b=d\ we\ get\ the\ next\ chain$

$$Similarly\ by\ substituting\ c=e\ \&\ d=f\ we\ get\ third\ chain$$

$$Putting\ d=g\ and\ f=h\ we\ get\ the\ fourth\ chain$$

Hence we get after putting values of (a,b,c,d,e,f,m) in equation ‘A’:

$2*(65)^2(x^2+y^2)^2=$

$$(-13y^2+13x^2-182xy)^2+\ (-91y^2+91x^2+26xy)^2$$
$$=\ (-89y^2+89x^2-46xy)^2+\ (-23y^2+23x^2+178xy)^2$$

$$=\ (-85y^2+85x^2-70xy)^2+\ (-35y^2+35x^2+170xy)^2$$

$$=(-79y^2+79x^2-94xy)^2+\ (-47y^2+47x^2+158xy)^2$$

For x=3,y =1 we get, after removing common factors,

$$2(325)^2=\ (221,403,)^2=\ (287,359)^2=(235,395)^2=(175{,}425)^2$$

The above chain can be made longer for bigger integers than w = 325 since they can be represented by more than four pairs of square numbers.

Since numerical solutions are not available where twice a kth power is equal to two kth powers for k=3,4 we have not attempted to parameterize it. Also it is to be noted that two kth powers have more than two chain lengths for k=3 &4 but are not equal to twice a kth power as required in this paper. Hence we have considered equations with three eelements in each chain. As given in the start of this paper our taxi cab equation (A) Is for k=2,3,4 & 5. We first paramatise for the 2nd degree & fourth degree for k=2 & 4. In this context we are aware that half a centaury ago M. Parasarathy and Albert Gloden gave a parametric solutions for 4-3-3 equation and their results included extending it for a long chain. The difference between their parameterization and this is that their paramatazisation did not sum up to twice a kth power & also our present method is less complicated.

The paramatized eqn. For K-2-3-3-3-3 is given below, for k=2 & k=4

We have from equation (A) above

Equation (2): $2(w)^k = (a,b,c)^k = (d,e,f)^k = (g,h,j)^k$=$(l,m,n)^k$

$$For\ k = 2\ \&\ 4$$

$$We\ have\ below\ the\ well\ known\ Identity: For\ k = 2\ \&\ 4$$

$$2(pq + q^2 + p^2)^k = (p^2 - q^2)^k + (-p^2 - 2pq)^k + (q^2 + 2pq)^k$$

$$Substituting\ p = r\ \&\ q = s, we\ get:$$

$$2(rs + r^2 + s^2)^k = (r^2 - s^2)^k + (r^2 + 2rs)^k + (s^2 + 2rs)^k$$

$$Hence\ we\ get;$$

$$2 = \frac{1}{(pq + q^2 + p^2)^k}\ ((p^2 - q^2)^k + (p^2 + 2pq)^k + (q^2 + 2pq)^k)$$

$$= \frac{1}{(rs + r^2 + s^2)^k}((r^2 - s^2)^k + (-r^2 - 2rs)^k + (s^2 + 2rs)^k)$$

$$Muliplying\ through\ out\ by\ \ (pq + q^2 + p^2)^k(rs + r^2 + s^2)^k$$

$We\ get\ the\ first\ two\ \ chains\ below$: Parametric form:

$$2(w)^k = (a,b,c)^k = (d,e,f)^k$$

w= $(pq + q^2 + p^2)(s^2 + r^2 + rs)$

---

a= $(s^2+r^2+rs)(p^2-q^2)$

b=$(s^2+r^2+rs)(q^2+2pq)$

c= $(s^2+r^2+rs)(-p^2-2pq)$

----

d=$(pq+q^2+p^2)(r^2-s^2)$

e=$(pq+q^2+p^2)(s^2+2rs)$

f=$(pq+q^2+p^2)(-r^2-2rs)$

The next chains are, $2(w)^k = (g,h,j)^k = (L,m,n)^k$

The third & fourth chains are given below:

Let, $(r^2-s^2) = u$, $(s^2+2rs) = v$, $(-r^2-2rs) = z$

$we\ have\ (u + v + z) = 0,$

hence we get after taking different combinations of

$$(q^2-2pq+p^2)(v+z+u)$$

$$\& \qquad (q^2-2pq+p^2)(u+v+z)$$

We get:

$$q^2(v+z+u)-2pq(z+u+v)+p^2(u+v+z)$$

Arranging this in a vertical matrix we get:

(say) g=$(q^2v-2pqz+p^2u)$

h=$(q^2z-2pqu+p^2v)$

j=$(q^2u-2pqv+p^2z)$

---

similarly we have :

$$q^2(u+v+z)-2pq(z+u+v)+p^2(v+z+u)$$

Arranging this in a vertical matrix we get:

(say) l= $(q^2u-2pqz+p^2v)$

m=$(q^2v-2pqu+p^2z)$

n =$(q^2z-2pqv+p^2u)$

$$Substituting\ for\ u,v,z\ \&\ factoring\ we\ get:$$

$$g=\ (2qr+pr+qs-ps)(pr+qs+ps)$$

$$h=(2qs+ps+2pr+qr)(ps-qr)$$

$$j=\ -(qs+2ps+pr-qr)(pr+qr+qs)$$

$$l=-(2pr+qr+ps-qs)(qr+ps+qs)$$

$$m=(2qr+pr+2ps+qs)(pr-qs)$$

$$n=(ps+2qs-pr+qr)(ps+pr+qr)$$

$$Acknowledgment:\ The\ fourth\ chain\ in\ the\ above\ was\ received$$

$$from\ Dr.Ajai\ Choudhry\ with\ thanks.$$

$$Below\ we\ show\ that\ \ the\ third\ \&\ fourth\ chain\ above,$$

$$2(w)^k=(g,h,j)^k\ ---Equation\ ‘1”$$

$$is\ true\ for\ \ p=q\ and\ k=2\ .\ We\ have$$

g=$(q^2v-2pqz+p^2u)$

h=$(q^2z-2pqu+p^2v)$

j=$(q^2u-2pqv+p^2z)$

We also have:

w = $(rs+r^2+s^2)(pq+q^2+p^2)$

$$For\ p=q\ ,\qquad we\ get,\qquad 2(w)^2=2*9*(q)^4(rs+r^2+s^2)^2$$

Right hand side of equation '1' above is:

we have:

$(r^2-s^2)=u,\quad (s^2+2rs)=v,\quad (-r^2-2rs)=z.$ We get:

$(g^2+h^2+j^2)=$

$$(q^2v-2pqz+p^2u)^2+(q^2z-2pqu+p^2v)^2+(q^2u-2pqv+p^2z)^2$$

$$=q^4((v-2z+u)^2+(z-2u+v)^2+(u-2v+z)^2)$$

$$=q^4\big(6(u^2+v^2+z^2)-6(uv+uz+vw)\big)$$

$$=6q^4\big((u^2+v^2+z^2)-(uv+uz+vz)\big)$$

$$=6q^4(-3(uv+uz+vz))\ ,[\,since\ u+v+z=0\ \&\ hence$$

$$((u^2+v^2+z^2)-(uv+uz+vz))=-3(uv+uz+vz)]$$

$$=-2*9(uv+uz+vz)$$

$$(Substituting\ values\ of\ u,v,z\ we\ get)$$

$$=-(2*9*q^4)(-)(s^4+r^4+3r^2s^2+2r^3s+2rs^3)$$

$$=(2)(9)(q^4)(rs+r^2+s^2)^2$$

Hence both sides of equation '1' are equal.

For other values of (p,q) a similar method can be followed for k=2,4

$$Hence\ For\ k = 2{,}4$$

$$2 * \left((p^2 + pq + q^2)(r^2 + rs + s^2)\right)^k$$

$$= (s^2 + r^2 + rs)^k((p^2 - q^2)^k + (q^2 + 2pq)^k + (-p^2 - 2pq)^k)$$

$$= (p^2 + q^2 + pq)^k((r^2 - s^2)^k + (s^2 + 2rs)^k + (-r^2 - 2rs)^k)$$

$$= \left((2qr + pr + qs - ps)(pr + qs + ps)\right)^k + \left((2qs + ps + 2pr + qr)(ps - qr)\right)^k + \left(-(qs + 2ps + pr - qr)(pr + qr + qs)\right)^k$$

$$= \left(-(2pr + qr + ps - qs)(qr + ps + qs)\right)^k + \left((2qr + pr + 2ps + qs)(pr - qs)\right)^k + \left((ps + 2qs - pr + qr)(ps + pr + qr)\right)^k$$

$$Numerical\ examples:$$

$$For\ (p, q, r, s) = (2{,}1{,}3{,}1)\ we\ get\ the\ four\ chains\ for\ k = 2\ \&\ 4$$

$$2(91)^k = (39{,}65{,}104)^k = (49{,}56{,}105)^k = (19{,}80{,}99)^k = (11{,}85{,}96)^k$$

$$For\ (p, q, r, s) = (4{,}3{,}5{,}2)\ , after\ taking\ out\ common\ factors\ we\ get$$

$$2(481)^k = (91{,}429{,}520)^k = (259{,}296{,}555)^k = (544{,}175{,}369)^k = (551{,}336{,}215)^k$$

Next we parameterize for power three, k=3 the cubic version, namely 3-2-3-3-3-3.

$$Consider\ the\ below\ mentioned\ equation,$$

$$2(m)^3 = (m + n)^3 + (m - n)^3 + (z)^3 - - - - - equation\ '1'$$

$$Let\ m = a^3b^3,\ \ n = 6p^3c^3$$

$Solving\ for\ \ z\ ,\qquad we\ get\qquad z = -6abc^2p^2$

$$Substituting\ values\ of\ m, n\ \&\ z\ in\ equation\ ‘1’\ we\ get$$

$2(a^3b^3)^3$
$= (a^3b^3 + 6p^3c^3)^3 + (a^3b^3 - 6p^3c^3\ )^\wedge 3 + (-6abc^2p^2)^3$

$$Substituting\ suitable\ values\ of\ p\ as$$
$$(q, r, s, ---) we\ get\ the\ chain\ equation.$$

Equation (A)
$2(a^3b^3)^3$

$$= (a^3b^3 + 6p^3c^3)^3 + (a^3b^3 - 6p^3c^3)^3 + (-6p^2abc^2)^3$$

$$= (a^3b^3 + 6q^3c^3)^3 + (a^3b^3 - 6p^3c^3)^3 + (-6q^2abc^2)^3$$

$$= (a^3b^3 + 6r^3c^3)^3 + (a^3b^3 - 6r^3c^3)^3 + (-6r^2abc^2)^3$$
$$= (a^3b^3 + 6s^3c^3)^3 + (a^3b^3 - 6s^3c^3)^3 + (-6s^2abc^2)^3$$

$$where\ (a, b, c)\ are\ parameters\ \&$$
$$(p, q, r, s, --)\ takes\ integer\ values$$

The above chain can be extended as long as needed by substituting different values for (x) in the equation $(a^3b^3 + 6x^3c^3)^3 + (a^3b^3 - 6x^3c^3)^3 + (-6x^2abc^2)^3$ and adding it to the existing chains in equation (3) above. Namely, ‘x’ represents (p,q,r,s,t,------) having integer values

The parametric form for 3-2-3-3-3-3, is given below:

$$We\ get: For\ p = 1, q = 2\ \&\ r = 3, s = 5, respectively,$$

$2(a^3b^3)^3$
$= (a^3b^3 + 6c^3)^3 + (a^3b^3 - 6c^3)^3 + (-6abc^2)^3$

$= (a^3b^3 + 48c^3)^3 + (a^3b^3 - 48c^3)^3 + (-24abc^2)^3$

$= (a^3b^3 + 162c^3)^3 + (a^3b^3 - 162c^3)^3 + (-54abc^2)^3$

$= (a^3b^3 + 750c^3)^3 + (a^3b^3 - 750c^3)^3 + (-150abc^2)^3$

Numerical example is: $for\ a = 3, b = 1,\ c = 1$

After removing common factors we get:

$$2(9)^3 = (11)^3 + (7)^3 + (-6)^3 = (25)^3 + (-7)^3 + (-24)^3$$
$$= (63)^3 + (-45)^3 + (-54)^3$$

$$= (259)^3 + (-241)^3 + (-150)^3$$

$$Next\ we\ parameterize\ for\ power\ five, k = 5\ the\ quintic,$$

$$(Note\ that, since\ there\ are\ no\ numerical\ solutions\ available\ for\ 5-2-3-3\ or\ 5-2-4-4\ or\ 5-2-5-5\ we\ selected\ 5-2-6-6\ to\ paramatise\ since\ numerical\ solutions\ are\ available\ for\ it.)$$

$$Consider\ equation\ (A), Given\ below,$$
$$(u,v,z) are\ parameters\ to\ be\ determined.$$

$2(t)^5(x^2+3y^2)^5 =$

$=(ax^2+3by^2+uxy)^5+(bx^2+3ay^2-uxy)^5+(cx^2+3dy^2+vxy)^5$

$+(dx^2+3cy^2-vxy)^5+(ex^2+3fy^2+zxy)^5+(fx^2+3ey^2-zxy)^5$

Where, $2(t)^5 = a^5 + b^5 + c^5 + d^5 + e^5 + f^5$

$$Where\ a+b=c+d=e+f=p$$
$$Hence\ a=p-b\ , c=p-d, e=p-f$$

We have numerical solutions like the one given below.

$$2(21)^5 = (13)^5 + (1)^5 + (-3)^5 + (17)^5 + (23)^5 + (-9)^5$$
$$Where\ (t,a,b,c,d,e,f,p) are\ known\ quantities,$$
$$Also, \quad p = a+b = 13+1, c+d = -3+17 = 14,$$
$$e+f = 23-9 = 14\ \&\ t = 21$$

$$As\ in\ the\ above\ example\ (t,a,b,c,d,e,f,p) = (21,13,1,-3,17,23,-9,14)$$

$$Expanding\ both\ sides\ of\ equation\ 'A'\ \&\ equating\ the\ coeeficents\ of\ 'x'\ we\ get\ after\ some\ algebra\ \ the\ equations\ below.$$

$$Substituting\ value\ of\ (a,c,e) in\ equation\ “A'\ and\ solving\ using\ Maple\ Math\ Software$$

$$\frac{v}{z} = \left(\frac{p^3 - 2p^2f + 3pf^2 - 2f^3 - 2p^2b + 3pb^2 - 2b^3}{-p^3 + 2p^2d - 3pd^2 + 2d^3 + 2p^2b - 3pb^2 + 2b^3}\right)$$

$$u = \left(\frac{p^3z - 4p^2zf + 6pzf^2 - 4zf^3 + p^3v - 4p^2vd + 6pvd^2 - 4vd^3}{-p^3 + 4p^2b - 6pb^2 + 4b^3}\right)$$

$$and \quad \frac{v}{z} = \left(\frac{b^4 - e^4 + f^4 - a^4}{a^4 + c^4 - b^4 - d^4}\right)$$

$Where \quad u = (v + z)$

$The\ three\ equations\ above\ are\ solved\ for\ the\ three\ variables\ (u, v, z)$

$Using\ the$ equation solve $feature\ in\ \ Maple\ Math\ Software\ we\ get\ the\ below$

$mentioned\ solutions\ for\ (u, v, z)$

$$u = 2(d - f)$$
$$v = 2(p - b - f)$$
$$z = 2(d + b - p)$$

$Also\ we\ have\ \ p = \ a + b = c + d = e + f$

$And\ p = 2(d - b + f)$

$We\ have\ initial\ solution:$

$2(3)^5 = (-1)^5 + (3)^5 + (-1)^5 + (3)^5 + (1)^5 + (1)^5$-------(G1) (say)

$Where\ (a, b, c, d, e, f, p)$
$= (-1,3,-1,3,1,1,2) we\ get\ after\ substituting\ in\ the\ equations\ for\ (u, v, z) above$
$we\ get\ t = 3, p = 2, u = 4, v = -4, z = 8$
$Thus\ after\ putting\ the\ values\ of\ \ (a, b, c, d, e, f, u, v, z, t)$

$2(3)^5(x^2 + 3y^2)^5 =$

$$(-x^2+9y^2+4xy)^5+(3x^2-3y^2-4xy)^5+(-x^2+9y^2-4xy)^5 + (3x^2-3y^2+4xy)^5+(x^2+3y^2+8xy)^5+(x^2+3y^2-8xy)^5$$

$$for\ x=2, y=1$$

$$We\ get, \qquad (x^2+3y^2)=7$$

$The\ result\ above\ \ is\ (say), (H1) = 2(3)^5(7)^5 = 2(21)^5 = (13,1,-3,17,23,-9)^5$

From above for $(a,b,c,d,e,f) = (13,1,-3,17,23,-9)$ & from equations for (u,v,z) above

$$and\ t=21\ p=14, u=52, v=44, \qquad z=8, \qquad p=a+b=14$$

we get equation,

$$2(21)^5(x^2+3y^2)^5 = (13x^2+3y^2+52xy)^5+(x^2+39y^2-52xy)^5 + (-3x^2+51y^2+44xy)^5+(17x^2-9y^2-44xy)^5 + (23x^2-27y^2+8xy)^5+(-9x^2+69y^2-8xy)^5$$

$$for\ x=2, y=1$$

$$The\ result\ is\ \ say\ (j1)\ = 2(21)^5(7)^5 = 2(147)^5 = (159,-61,127,-29,81,17)^5$$

$$From\ above\ we\ get\ for\ \ (a,b,c,d,e,f) = (159,-61,127,-29,81,17)$$

$$and\ t=147, p=98, u=\ -92, v=284, \qquad z=\ -376$$

$$we\ get\ for\ x=2, y=1$$

$$2(147)^5(x^2+3y^2)^5 = (159x^2-183y^2-92xy)^5+(-61x^2+477y^2+92xy)^5 + (127x^2-87y^2+284xy)^5+(-29x^2+381y^2-284xy)^5 + (81x^2+51y^2-376xy)^5+(17x^2+243y^2+376xy)^5$$

For above we get for x=2,y=1, $(a,b,c,d,e,f) = (269,417,989,-303,-377,1063)$,

$$t=1029$$

$$The\ result\ is\ \ say\ (L1) =\ 2(1029)^5 = 2*(147)^5*(7)^5$$

and $t=1029, p=688, u=-2732, v=-1588,\ z=-1144$

$$2(1029)^5(x^2+3y^2)^5 = (269x^2+1251y^2-2732xy)^5 + (417x^2+807y^2+2732xy)^5 + (989x^2-909y^2-1588xy)^5 + (-303x^2+2967y^2+1588xy)^5 + (-377x^2+3189y^2-1144xy)^5 + (1063x^2-1131y^2+1144xy)^5$$

Taking the Least common multiple for the four chains we get the below equation:

$$2*(1029)^5 = (343)^5*(G1) = (49)^5*(H1) = (7)^5*(J1) = (L1)$$

Substituting for $(G1),(H1),(J1),(L1)$ *from equations above, we get the required taxicab chain., which can be extended as long as needed*)

The paramatized fifth power equation is given below:

$$2*(w)^5 = (m_1,m_{2,}m_{3,}m_{4,}m_{5,}m_6)\^{}5 = (n_{1,},n_{2,}n_{3,}n_{4,}n_5,n_{6,})\^{}5 = (p_{1,}p_{2,}p_{3,}p_{4,}p_{5,}p_6)\^{}5 = (q_{1,}q_{2,}q_{3,}q_{4,}q_{5,}q_6)\^{}5$$

For $w = 1029(x^2+3y^2)$ *, we get the four chains below,*

$$2*(1029)^5(x^2+3y^2)^5 = 343^5((-x^2+9y^2+4xy)^5 + (3x^2-3y^2-4xy)^5 + (-x^2+9y^2-4xy)^5 + (3x^2-3y^2+4xy)^5 + (x^2+3y^2+8xy)^5 + (x^2+3y^2-8xy)^5)$$

$$= 49^5((13x^2+3y^2+52xy)^5 + (x^2+39y^2-52xy)^5 + (-3x^2+51y^2+44xy)^5 + (17x^2-9y^2-44xy)^5 + (23x^2-27y^2+8xy)^5 + (-9x^2+69y^2-8xy)^5)$$

$$= 7^5(159x^2-183y^2-92xy)^5 + (-61x^2+477y^2+92xy)^5 + (127x^2-87y^2+284xy)^5 + (-29x^2+381y^2-284xy)^5 + (81x^2+51y^2-376xy)^5 + (17x^2+243y^2+376xy)^5$$

$$= (269x^2 + 1251y^2 - 2732xy)^5 + (417x^2 + 807y^2 + 2732xy)^5$$
$$+ (989x^2 - 909y^2 - 1588xy)^5$$
$$+ (-303x^2 + 2967y^2 + 1588xy)^5$$
$$+ (-377x^2 + 3189y^2 - 1144xy)^5$$
$$+ (1063x^2 - 1131y^2 + 1144xy)^5$$

Numerical example is: For x=1 & y=0, we get, k=5

$$2(w)^k = 2 * (1029)^5$$

$$= (343)^5 * (-1,3,-1,3,1,1)^5$$

$$= (49)^5 * (13,1,-3,17,23,-9)^5$$

$$= (7)^5 * (17,81,-29,127,-61,159)^5$$

$$= (269,417,989,-303,-377,1063)^5$$

<u>*Summary*</u>:

In conclusion we note that for powers k=2,4For even powers the maximum chain length can be extended beyond four chain lengths. $For\ larger\ values\ of\ w >\ 91$.

$$where \quad 2(w)^k = (a^k + b^k + (a+b)^k)$$

$$Example\ for\ k = 2, 4$$

$$2 * (1729)^k = (931,1064,1995)^k = (845,1144,1989)^k$$

$$= (799,1185,1984)^k = (741,1235,1976)^k$$

$$= (656,1305,1961)^k$$

This above numerical solution can be paramatised & we can arrive at nine chains, but it involves using six variables instead of four variables (p,q,r,s) used above & the equations do not have a simple form. Alternatively another approach might help paramatise longer chains of length more than four. For k= 3 & 5 odd powers our taxicab equation (A) can have as long chains as needed since the terms in the chain are signed, meaning the terms take on plus & minus values. But k=5, remains an open problem for taxicab equation (A) above in the sense that numerical results are awaited in which paramatization of chains having less than six terms can be attempted. Numerical solutions are available for (5-4-4) but they are not equal to twice a fifth power as required in this paper. Beyond k=5, we have Euler's conjecture to contend with, but with computers becoming more powerful, maybe in the future we will have numerical solutions which might help us paramatize them.

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